On the vertical profile of velocity in 3D SELFE model: open channel flow down a slope

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While the physics for this problem can be quite complex (e.g. involving some inner works of k-ε closure) we focus on a simple analytical solution (Chezy flow), which is often used to derive the Rouse profile for sediment concentration, and use it to rigorously validate SELFE numerical and physical formulations.

Assuming at the steady state the flow is uniform (Fig. 1) and the free surface is parallel to the bottom, the momentum balance is given by:

$$\begin{cases} \dfrac{\partial}{\partial z}\left(\nu \dfrac{\partial u}{\partial z}\right) - g\eta_x = 0 \\ \nu \dfrac{\partial u}{\partial z} = \tau_b, \ z = -h \\ u = 0, \ z = -h \\ \nu \dfrac{\partial u}{\partial z} = 0, \ z = \eta \end{cases}, \qquad (1)$$

where u is the horizontal velocity, η is the elevation, and ν is the viscosity, τ_b is the bottom stress (typically evaluated at the top of bottom cell in 3D models). Integrating the momentum eq. over depth gives the 2D balance (Chezy):

$$\tau_b = -g\eta_x = g\gamma H = (const) \qquad (2)$$

From which the depth-averaged velocity and total flow can be computed:

$$\tau_b = \hat{C}_d \hat{u}^2,$$
$$\hat{u} = \left(\frac{g\gamma H}{\hat{C}_d} \right)^{1/2} \qquad (3)$$
$$Q = \hat{u} H W$$

where $\hat{C}_d$ is the drag coefficient for *2D* model, *H* is the total depth, and *W* is the channel width.

Let's look at a lab set-up (suggested by Dr. Aron Roland) as depicted in Fig. 1: *L*=40m, *W*=4m, *H*=2m, γ=2.e-4, $\hat{C}_d$ =0.01. Then from the analytical solution we have $\hat{u}$ =0.624m/s, *Q*=5.01m^3/s. For SELFE set-up, Δx=Δy=1m, Δt=60s, and the total run time is 0.1 days with the steady state established quickly. A constant discharge is imposed at the left bnd and the elevation is clamped at the analytical value at the right bnd. Fig. 2 shows the comparison.

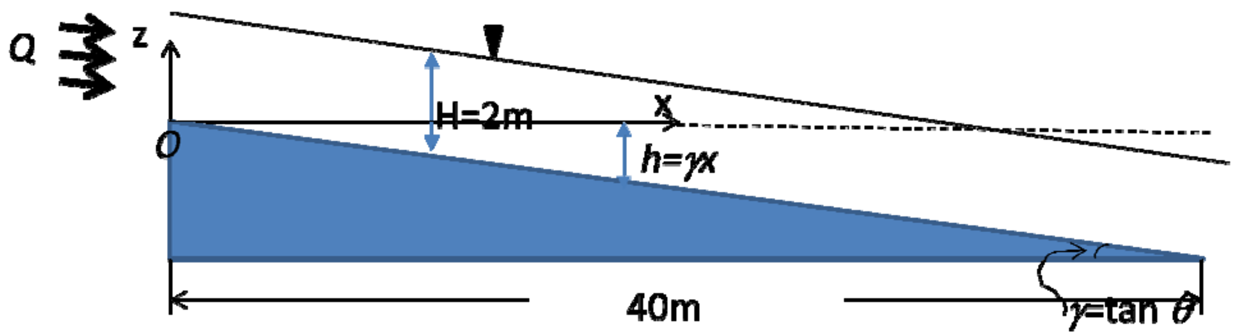

Fig. 1: Open channel flow down a slope: definition sketch.

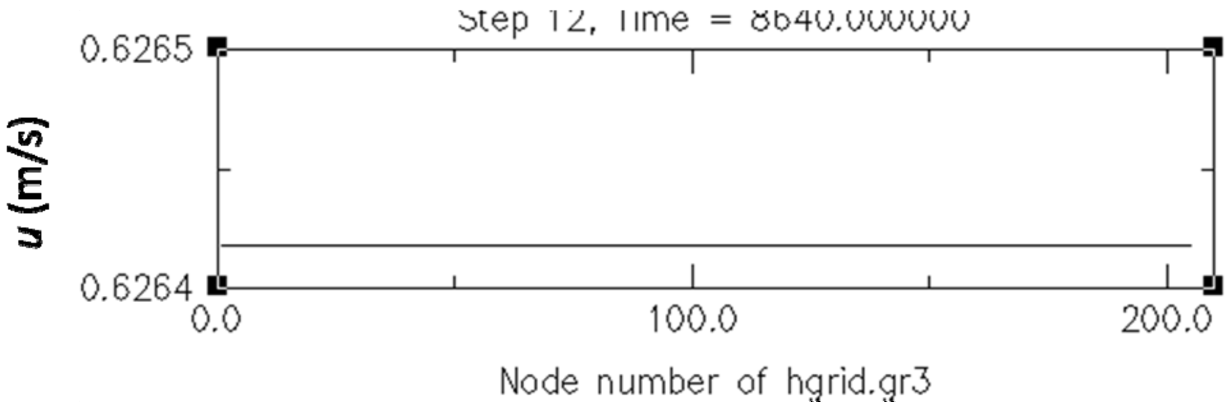

Fig. 2: Modeled depth-averaged velocity at all nodes at the end of 0.1-day run; the analytical value is indistinguishable from model results. $\hat{C}_d$ =0.01.

3D solution

Integrating Eq. (1) once and applying the surface b.c. give:

$$\nu \frac{\partial u}{\partial z} = g\gamma(\eta - z) \qquad (4)$$

It's more convenient to work with a transformed vertical coordinate – the distance from bed:

$$\tilde{z} = z + h \quad (\geq 0) \qquad (5)$$

And then

$$\nu \frac{\partial u}{\partial \tilde{z}} = g\gamma(H - \tilde{z}) \qquad (6)$$

Note that Eq. (6) ensures the bottom stress condition in Eq. (1) is satisfied (at $\tilde{z}$ =0; cf. Eq. (2)). The remaining no-slip b.c. is needed below.

Now assuming a quadratic profile for the viscosity:

$$\nu = \lambda(H - \tilde{z})(z_1 + \tilde{z}) \qquad (7)$$

Where z_1=1mm<<H is a small constant (can be thought as related to roughness but not same) needed to avoid singularity at bottom, and λ is a scaling constant to ensure consistency between

2D and 3D solutions. Note that the maximum viscosity occurs at $\tilde{z} = (H - z_1)/2$ or roughly at the middle of the column, and ν=0 at the surface. The choice of viscosity is not unique but the solution here is simpler in form.

Integrating (6) and (7) and applying the no-slip b.c. lead to:

$$u(\tilde{z}) = \frac{g\gamma}{\lambda} \log\left(\frac{z_1 + \tilde{z}}{z_1}\right) \tag{8}$$

It may look odd that *u* is inversely proportional to λ, but this makes sense because as λ →0, ν →0 and inviscid flow does not experience bottom friction and so the bottom is effectively frictionless ($\hat{C}_d$ →0) and so $\hat{u}$ →infinity; in other words without the energy loss due to friction there will be no balance in the first place!

Finally the scaling constant λ is determined by consistency between 2D and 3D solns:

$$\hat{u} = \frac{1}{H}\int_0^H u(\tilde{z})d\tilde{z} \tag{9}$$

And so

$$\lambda = \frac{g\gamma}{\hat{u}}\left[(\log(H/z_1) - 1)\right] \qquad (z_1 << H) \tag{10}$$

which has a dimension of s^{-1}.

In 3D models the drag coefficient C_d (not to be confused with $\hat{C}_d$) is specified directly or from the bottom roughness, and therefore we need to calculate its analytical value that is consistent with the formulation above as:

$$\tau_b = C_d u^2(\tilde{z}_2),$$

$$C_d = \frac{g\gamma H}{u^2(\tilde{z}_2)} \tag{11}$$

where $\tilde{z} = \tilde{z}_2$ is the coordinate at the 2nd lowest level. Eq. (11) can be used to calculated the bottom roughness if needed.

3D model

Using the parameters above we find $\lambda = \nu_{max}$=0.02, and the drag coefficient depends on the vertical grid used (we used 13 and 25 levels (Fig. 3), the latter doubling the resolution of the former):

$$C_d = 0.020 - 0.028$$

Note that the drag coefficient is larger than 2D case to ensure the bottom stress stays the same.

The 3D model requires a smaller time step (20s) to get reasonable results. Also there is some adjustment near the left bnd which leads to less accurate results there (cf. Fig. 4a; top panel); therefore the 3D flow is not as uniform as 2D. Overall SELFE seems to be doing a reasonable job in simulating the log profile thru'out the domain; as more levels are used the bottom boundary layer is also accurately calculated (Fig. 4b).

Remarks on Warner (2005) test

Compared to Warner et al. (2005), the test case used here has a steep slope and smaller scale. The Δx and Δt etc. used here can be scaled up to do that test. However, the most important difference lies in the fact that in that paper there is no comparison against a true analytical soln like here; comparisons were simply made between different profiles of viscosity (including one with parabolic form). Therefore in my view it's not a rigorous validation of model.

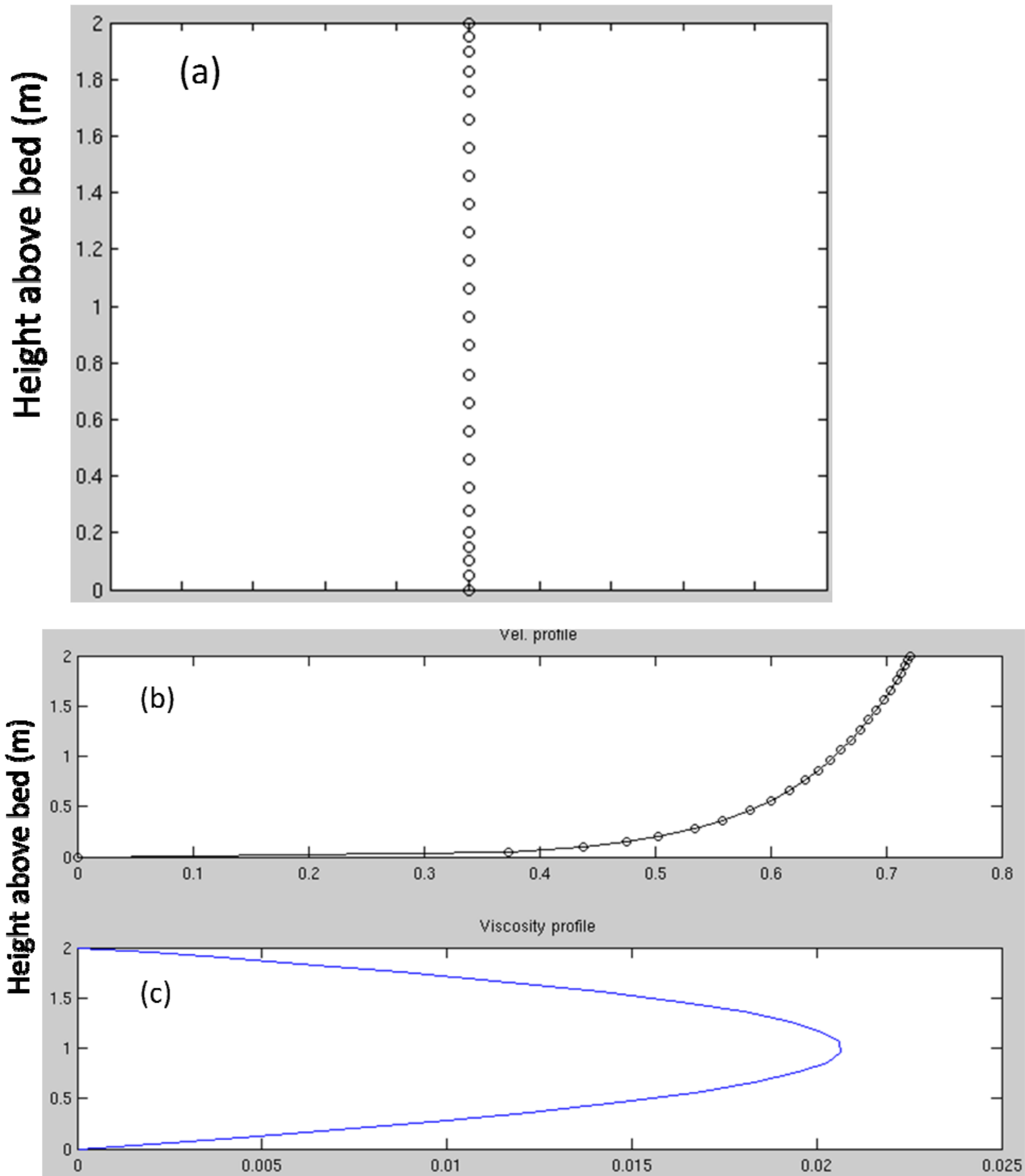

Fig. 3: Analytical soln. (a) Vertical grid with 25 σ levels. Since the total depth in this case is below the threshold set inside SELFE for *S* coordinates we cannot use *S* here but attempted to adjust σ levels to mimic *S*. In practical applications *S* should be used. (b) velocity profile; (c) viscosity profile.

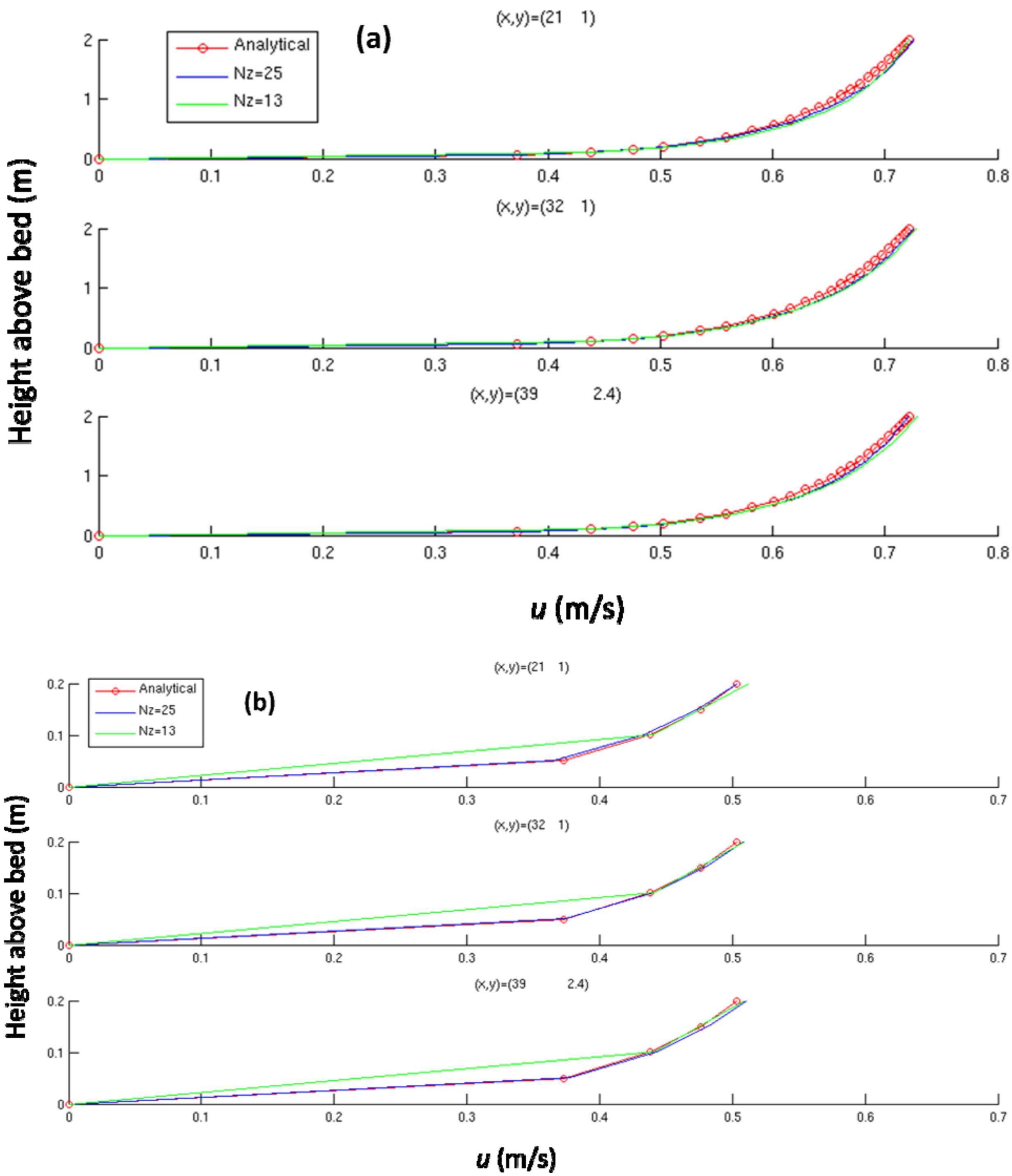

Fig. 4: Comparison of steady-state velocity profiles at 3 points. (a) full profile; (b) zoom-in near bottom. We used indvel=1 in these runs but indvel=0 gave similar results.