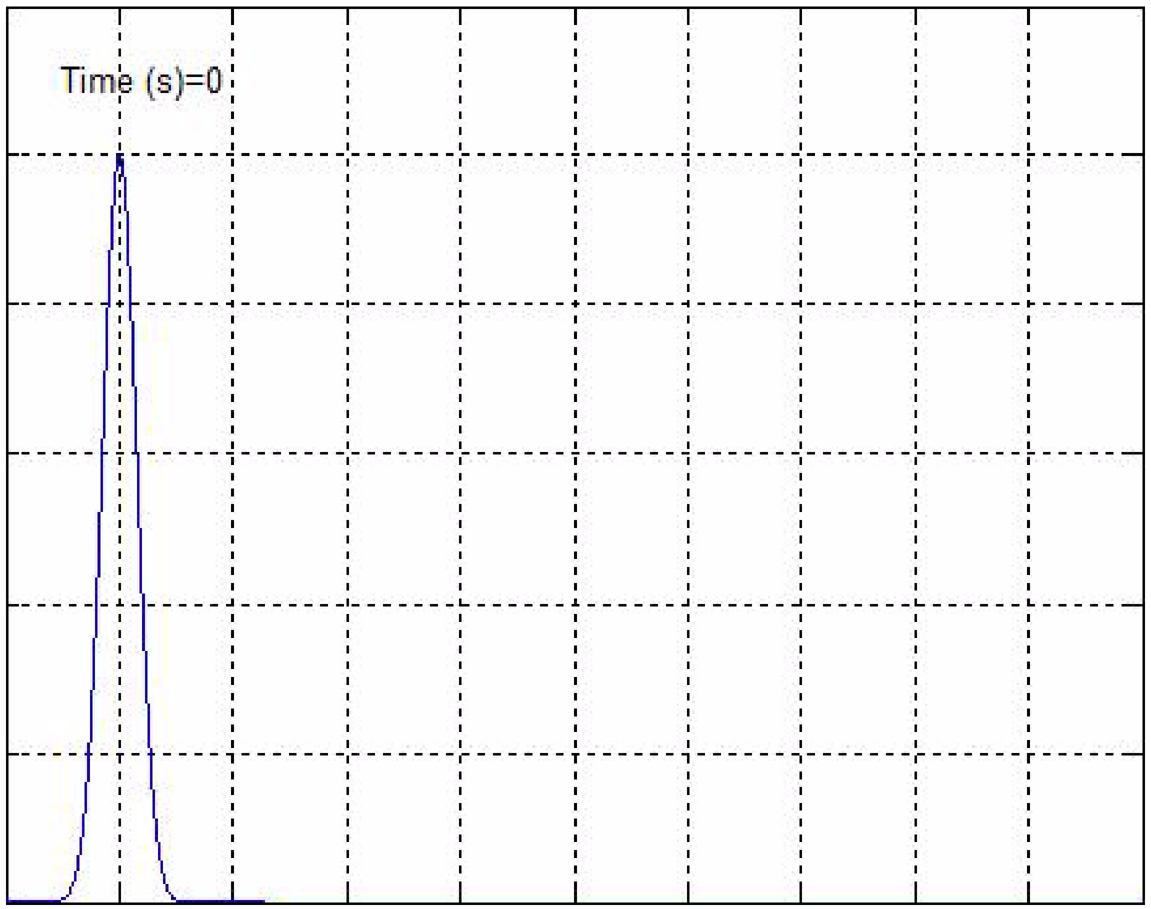
SCHISM physical formulation

Joseph Zhang

Advection

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + \boldsymbol{u} \cdot \nabla c = 0$$

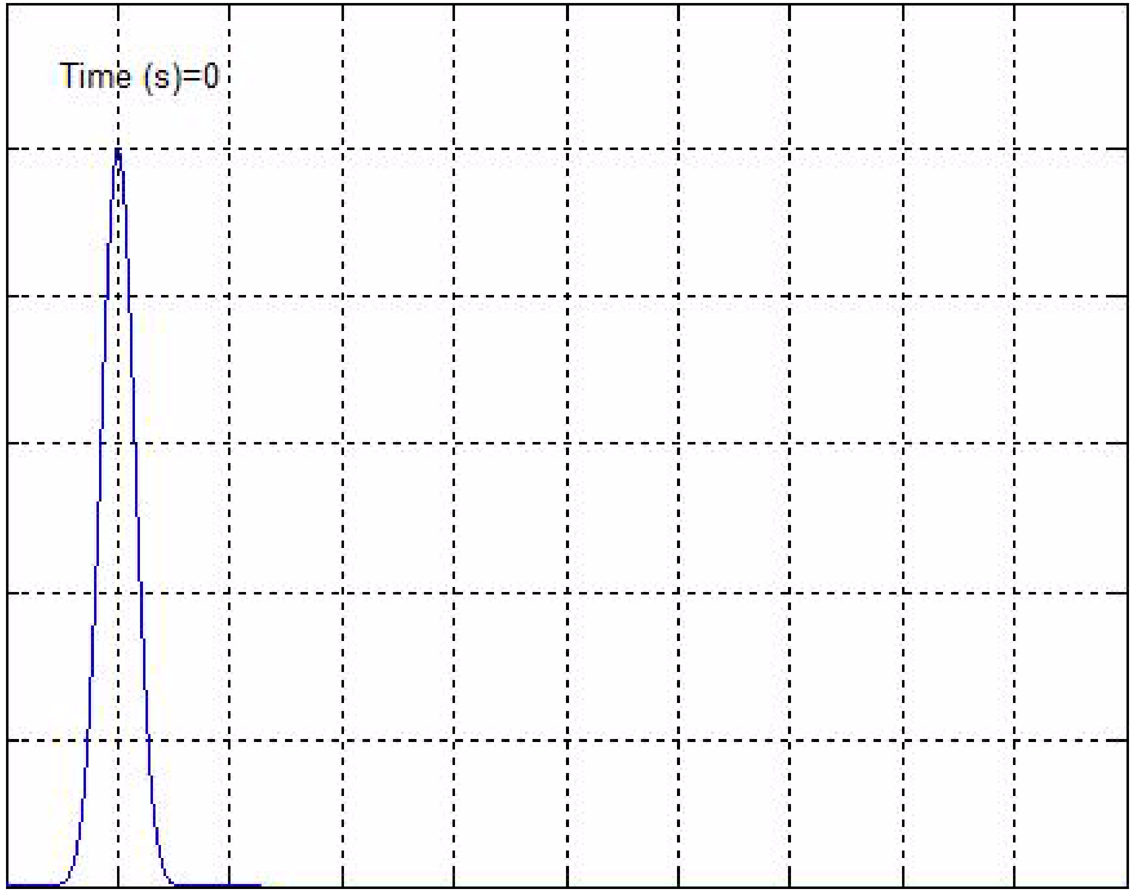
$\boldsymbol{u}$: advective velocity (usually known)



Diffusion

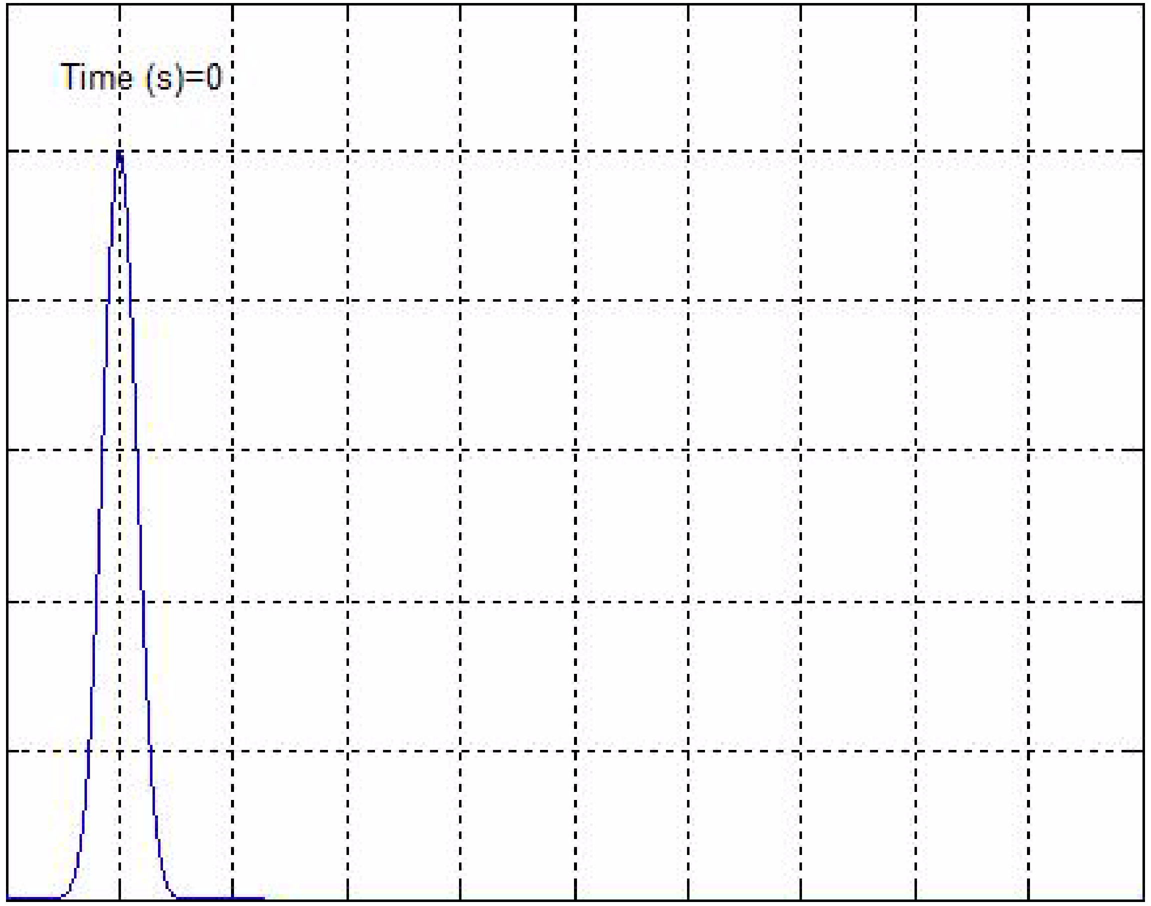
$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial z}\left(\kappa \frac{\partial}{\partial z}\right)$$

κ : diffusivity [m^2/s]

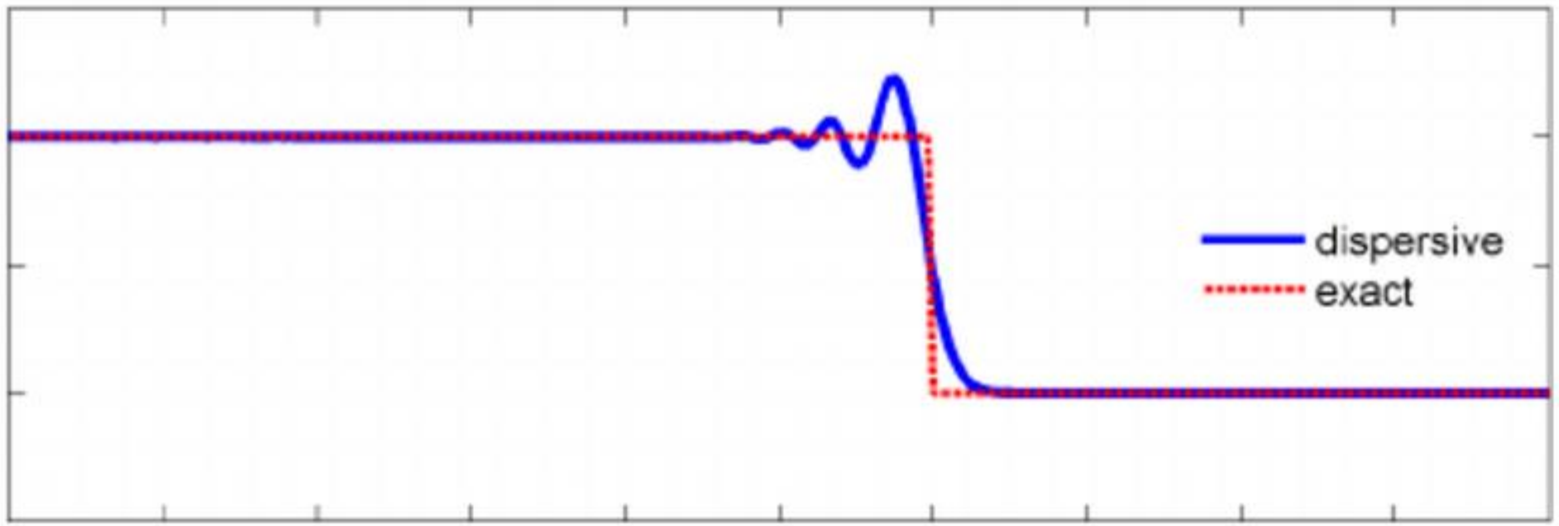


Advection-Diffusion

$$\frac{Dc}{Dt} = \frac{\partial}{\partial z}\left(\kappa \frac{\partial}{\partial z}\right)$$

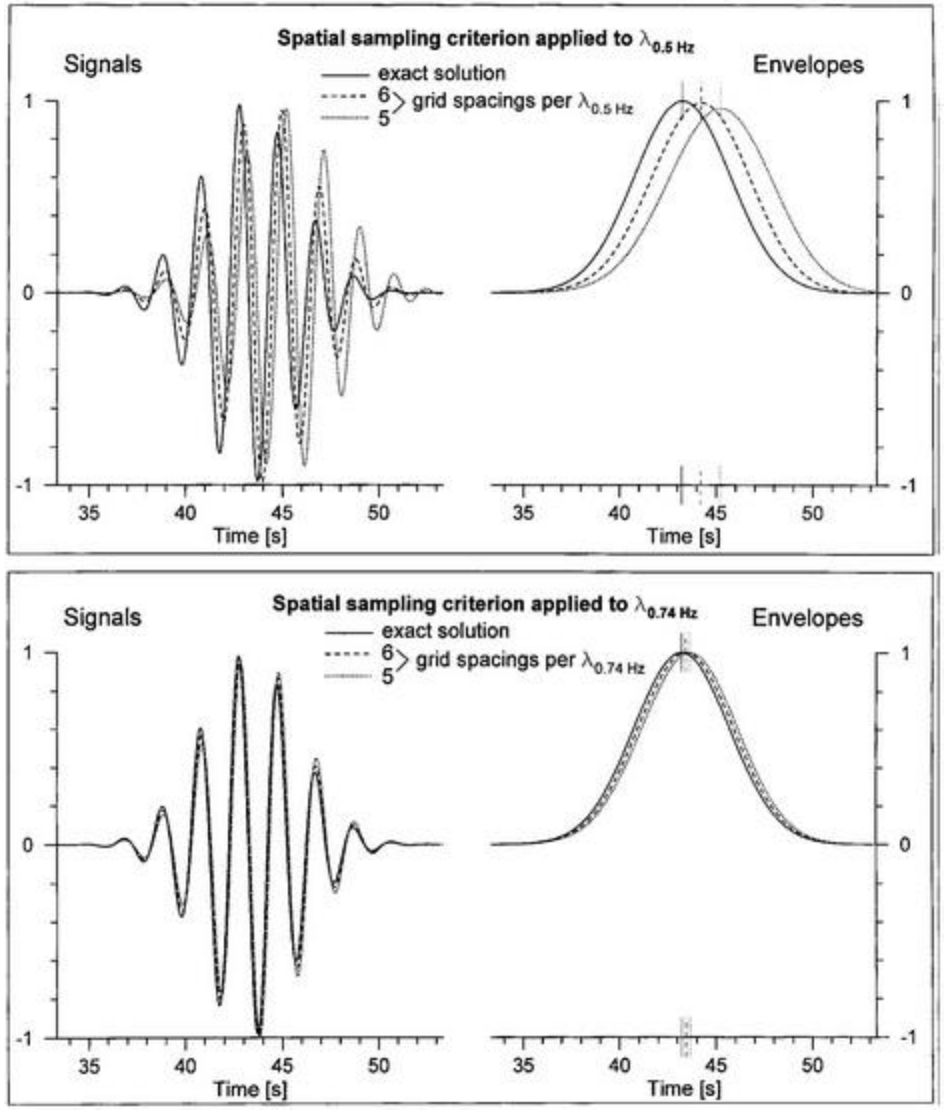


Dispersion



- Waves of different frequencies travel at different phase speeds
- Usually related to derivatives of odd order (of either physical or numerical origin)

$$\varphi_t + \varphi_{xxx} - 6\varphi\varphi_x = 0$$



With dispersion No dispersion

Governing equations: Reynolds-averged Navier-Stokes (with vegetation)

- Continuity equation

$$\nabla\cdot\mathbf{u}+\frac{\partial w}{\partial z}=0,\ \ (\mathbf{u}=(u,v))$$

$$\frac{\partial \eta}{\partial t}+\nabla\cdot\int_{-h}^{\eta}\mathbf{u}dz=0$$

- Momentum equations

vegetation

$$\frac{D\boldsymbol{u}}{dt}=\mathbf{f}-g\nabla\eta+\boldsymbol{m}_{\boldsymbol{z}}-\alpha|\boldsymbol{u}|\boldsymbol{u}L(x,y,z)\qquad \mathbf{f}=-f\mathbf{k}\times\mathbf{u}+\alpha g\nabla\hat{\psi}-\frac{1}{\rho_0}\nabla p_A-\frac{g}{\rho_0}\int_z^{\eta}\nabla\rho d\zeta+\nabla\cdot(\mu\nabla\mathbf{u})$$

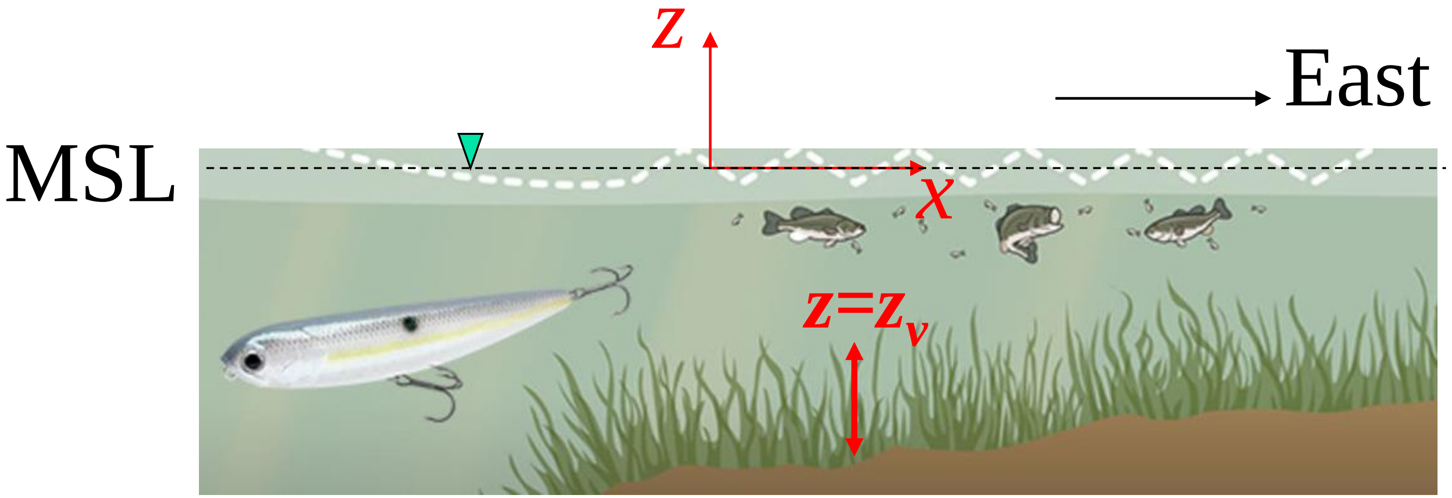
Vertical b.c. (3D):

$$\begin{cases}\nu\dfrac{\partial\boldsymbol{u}}{\partial z}=\boldsymbol{\tau}_w, & z=\eta\\ \nu\dfrac{\partial\boldsymbol{u}}{\partial z}=\chi\boldsymbol{u}_b, & z=-h\end{cases}$$

$$\boldsymbol{m}_z=\begin{cases}\dfrac{\partial}{\partial z}\left(\nu\dfrac{\partial\boldsymbol{u}}{\partial z}\right), & 3D\\ \dfrac{\boldsymbol{\tau}_w-\chi\boldsymbol{u}}{H}, & 2D\end{cases}$$

$$L(x,y,z)=\begin{cases}\mathcal{H}(z_v-z), & 3D\\ 1, & 2D\end{cases}$$

$\alpha(x,y)=D_vN_vC_{Dv}/2$ (vegetation)



Governing equations: Reynolds-averged Navier-Stokes (with SAV)

- Equation of state $\rho = \rho(p, S, T)$

- Transport of salt and temperature

$$\frac{Dc}{Dt} = \frac{\partial}{\partial z}\left(\kappa \frac{\partial c}{\partial z}\right) + \dot{Q} + \nabla \cdot (\kappa_h \nabla c), \qquad c = (S, T)$$

- Turbulence closure: Umlauf and Burchard 2003

shear stratification

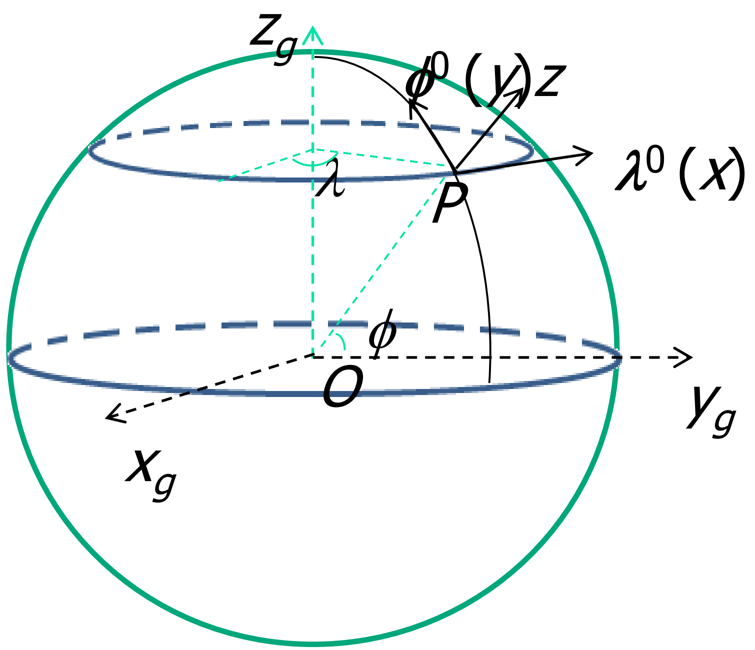
$$\frac{Dk}{Dt} = \frac{\partial}{\partial z}\left(\nu_k^{\psi} \frac{\partial k}{\partial z}\right) + \nu M^2 + \kappa N^2 - \epsilon + c_{fk}\alpha|\boldsymbol{u}|^3\mathcal{H}(z_v - z)$$

vegetation

$$\frac{D\psi}{Dt} = \frac{\partial}{\partial z}\left(\nu_{\psi} \frac{\partial \psi}{\partial z}\right) + \frac{\psi}{k}\left[c_{\psi 1}\nu M^2 + c_{\psi 3}\kappa N^2 - c_{\psi 2}\epsilon F_{wall} + c_{f\psi}\alpha|\boldsymbol{u}|^3\mathcal{H}(z_v - z)\right]$$

$$\psi = \left(c_{\mu}^{0}\right)^{p} k^{m} \ell^{n},$$

On spherical coordinates



- Avoids polar singularity easily
- Preserves all original matrix properties

$$\begin{cases} x_g = R_0 \cos\phi \cos\lambda \\ y_g = R_0 \cos\phi \sin\lambda \\ z_g = R_0 \sin\phi \end{cases}$$

$$\boldsymbol{\lambda}^0 = -\sin\lambda \boldsymbol{i} + \cos\lambda \boldsymbol{j}$$
$$\boldsymbol{\phi}^0 = -\cos\lambda \sin\phi \boldsymbol{i} - \sin\lambda \sin\phi \boldsymbol{j} + \cos\phi \boldsymbol{k}$$

We transform the coordinates instead of eqs
There are 2 frames used:

1. Global
2. Lon/lat (local @ node/element/side)

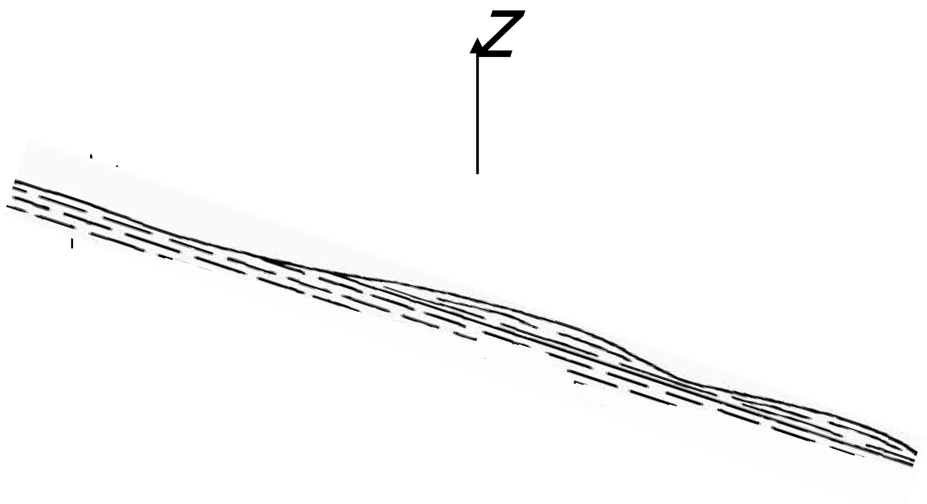
Changes to the code is mainly related to re-project vectors and to calculate distances

Continuity equation

Vertical boundary conditions: kinematic

surface

$$\begin{cases} z=\eta(x,y,t) \\ w\equiv \dfrac{dz}{dt}=\dfrac{dx}{dt}\dfrac{\partial\eta}{\partial x}+\dfrac{dy}{dt}\dfrac{\partial\eta}{\partial y}+\dfrac{\partial\eta}{\partial t}=u\eta_x+v\eta_y+\eta_t \end{cases}$$



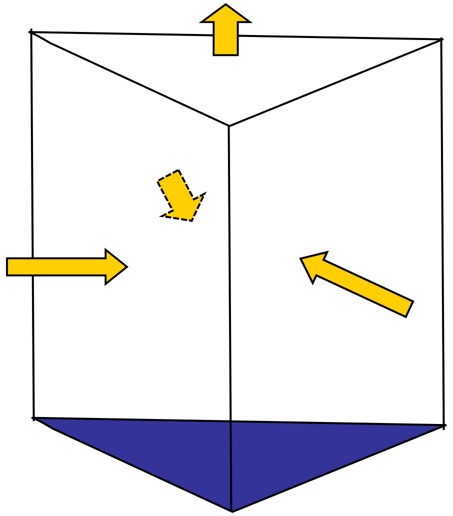
bottom

$$\begin{cases} z=-h(x,y) \\ w\equiv \dfrac{dz}{dt}=-uh_x-vh_y \end{cases}$$

$$\int_{-h}^{\eta}\nabla\cdot\mathbf{u}dz+w|_{-h}^{\eta}=0$$

Integrated continuity equation

$$\frac{\partial\eta}{\partial t}+\nabla\cdot\int_{-h}^{\eta}\mathbf{u}dz=0$$



$$\int_{-h}^{\eta}\nabla\cdot\mathbf{u}dz+\eta_t+u\eta_x+v\eta_y+uh_x+vh_y=0$$

$$\nabla\cdot\int_{-h}^{\eta}\mathbf{u}dz=\int_{-h}^{\eta}\nabla\cdot\mathbf{u}dz+\mathbf{u}\cdot\nabla(\eta+h)$$

Hydrostatic model

Hydrostatic assumption

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} - g = 0 \Longrightarrow P = g\int_z^{\eta} \rho \, d\zeta + P_A$$

Separation of horizontal and vertical scales

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla\left(\int_z^{\eta} \rho \, d\zeta + P_A\right) + \nabla\cdot(\mu\nabla\mathbf{u}) + \frac{\partial}{\partial z}\left(\nu\frac{\partial \mathbf{u}}{\partial z}\right) - f\mathbf{k}\times\mathbf{u}$$

$$\frac{D\mathbf{u}}{Dt} = -g\nabla\eta + \frac{\partial}{\partial z}\left(\nu\frac{\partial \mathbf{u}}{\partial z}\right) - f\mathbf{k}\times\mathbf{u} + \alpha g\nabla\hat{\psi} - \frac{1}{\rho_0}\nabla p_A - \frac{g}{\rho_0}\int_z^{\eta}\nabla\rho \, d\zeta + \nabla\square(\mu\nabla\mathbf{u})$$

Boussinesq assumption (=>incompressibility)

$$\nabla\int_z^{\eta}\rho \, d\zeta = \int_z^{\eta}\nabla\rho \, d\zeta + \rho\nabla\eta$$

Momentum equation: vertical boundary condition (b.c.)

Surface $\nu \dfrac{\partial \mathbf{u}}{\partial z} = \boldsymbol{\tau}_w$ at $z=\eta$ Bottom $\nu \dfrac{\partial \mathbf{u}}{\partial z} = C_D \,|\mathbf{u}_b|\, \mathbf{u}_b$, at $z=-h$

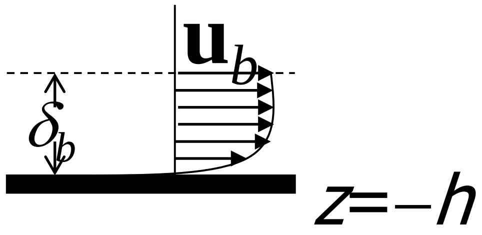
- Logarithmic law:

$$\mathbf{u} = \frac{\ln[(z+h)/z_0]}{\ln(\delta_b/z_0)}\mathbf{u}_b, \quad (z_0 - h \le z \le \delta_b - h)$$

z_0 : bottom roughness

- Reynolds stress:

$$\nu \frac{\partial \mathbf{u}}{\partial z} = \frac{\nu}{(z+h)\ln(\delta_b/z_0)}\mathbf{u}_b$$



- Turbulence closure:

$$\nu = \sqrt{2} s_m K^{1/2} l,$$
$$s_m = g_2,$$
$$K = \frac{1}{2} B_1^{2/3} C_D \,|\mathbf{u}_b|^2$$
$$l = \kappa_0 (z+h)$$

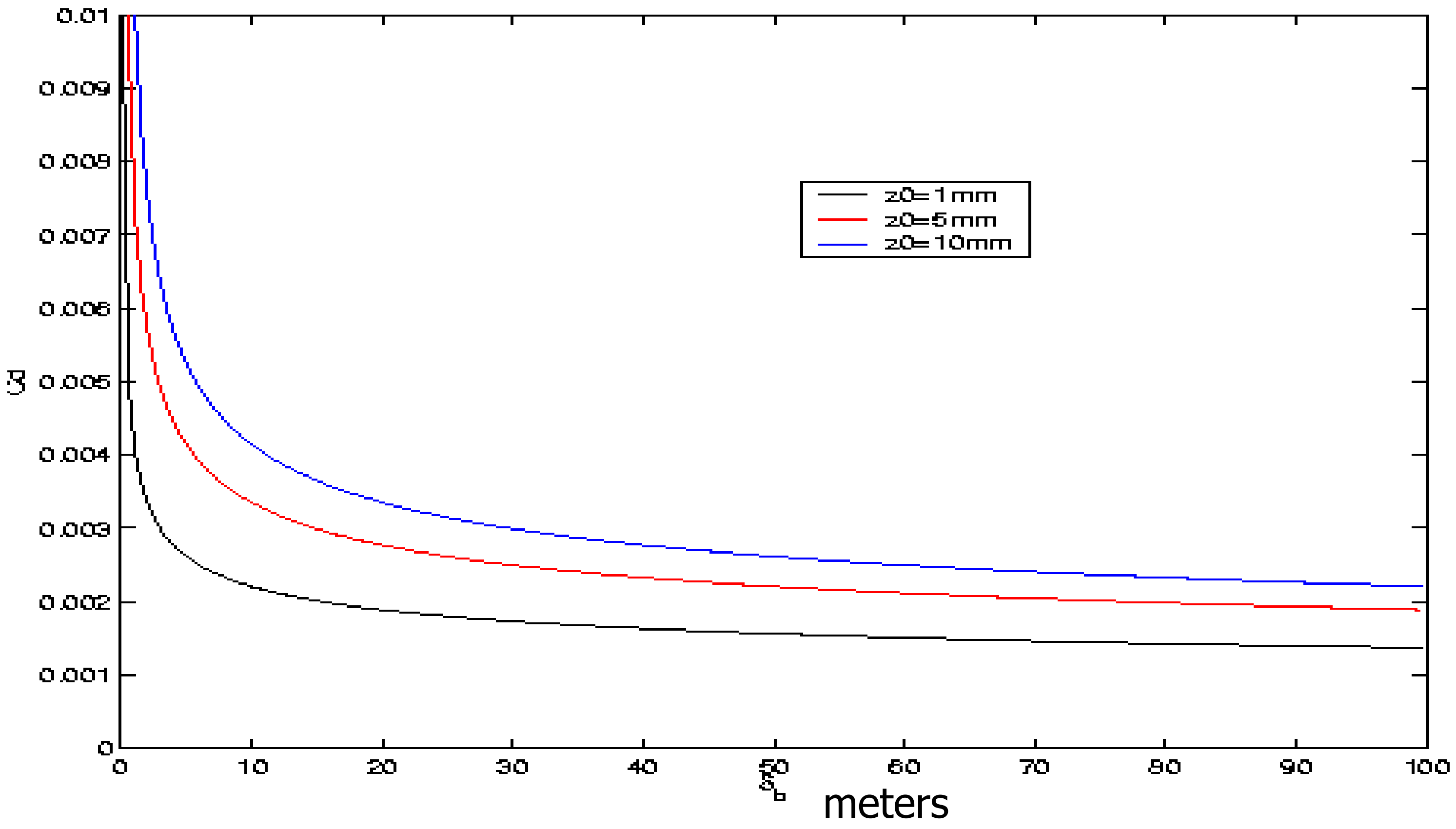
- Reynolds stress (const.)

$$\nu \frac{\partial \mathbf{u}}{\partial z} = \frac{\kappa_0}{\ln(\delta_b/z_0)} C_D^{1/2} \,|\mathbf{u}_b|\, \mathbf{u}_b, \quad (z_0 - h \le z \le \delta_b - h)$$

- Drag coefficient:

$$C_D = \left(\frac{1}{\kappa_0}\ln\frac{\delta_b}{z_0}\right)^{-2}$$

Bottom drag



- Use of z_0 seems most natural option, but tends to over-estimate C_D in shallow area
- C_D should vary with bottom layer thickness (i.e. vertical grid)
- Different from 2D, 3D results of elevation depend on vertical grid (with constant C_D)

Transport equation: source terms

Precipitation and evaporation model

$$\kappa \frac{\partial S}{\partial z} = \frac{S(E-P)}{\rho_0}, \quad z = \eta$$

E: evaporation rate (kg/m^2/s) (calculated from heat exchange model)

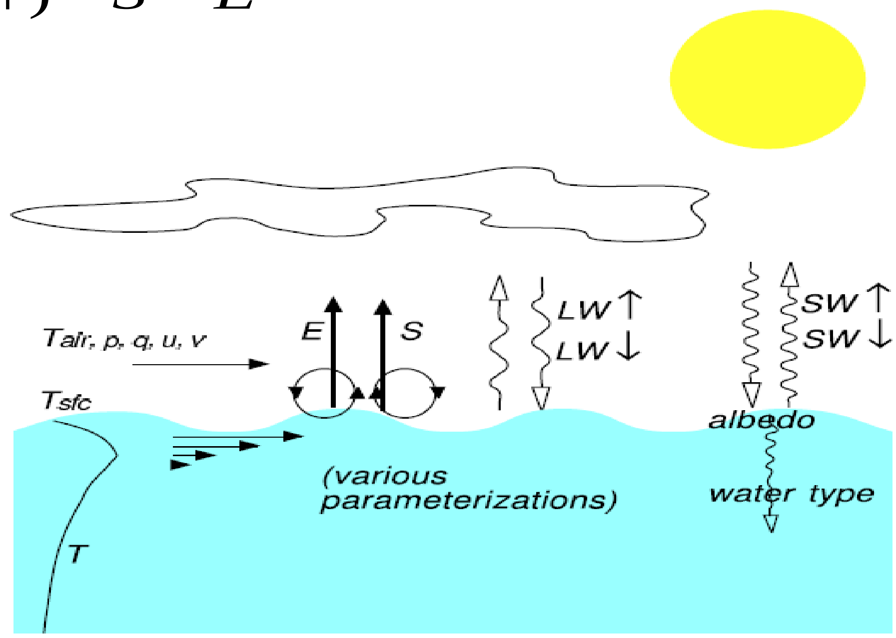
P: Precipitation rate (kg/m2/s) (measured)

Heat exchange model

$$\kappa \frac{\partial T}{\partial z} = \frac{\dot{H}}{\rho_0 C_p}, \quad z = \eta$$

$$\dot{Q} = \frac{1}{\rho_0 C_p} \frac{\partial SW}{\partial z}$$

$$\dot{H} = (IR\downarrow - IR\uparrow) - S - E$$



IR's are down/upwelling infrared (LW) radiation at surface
S is the turbulent flux of sensible heat (upwelling)
E is the turbulent flux of latent heat (upwelling)
SW is net downward solar radiation

The solar radiation is penetrative (body force) with attenuation (which depends upon turbidity) acts as a heat source within the water.

Air-sea exchange

1. *Momentum:* near-surface winds apply wind-stress on surface (influences advection, location of density fronts)
   - *free surface height:* variations in atmospheric pressure over the domain have a direct impact upon free surface height; set up due to wind stresses
2. *Heat:* various components of heat fluxes (dependent upon many variables) determine surface heat budget
   - shortwave radiation (solar) - penetrative
   - longwave radiation (infrared)
   - sensible heat flux (direct transfer of heat)
   - latent heat flux (heating/cooling associated with condensation/evaporation)
3. *Mass:* evaporation, condensation, precipitation act as sources/sinks of fresh water

1-2 are accomplished thru the heat/salt exchange model

Solar radiation

Downwelling *SW* at the surface is forecast in NWP models - a function of time of year, time of day, weather conditions, latitude, etc [sflux_rad*.nc]
Upwelling *SW* is a simple function of downwelling *SW*

$$SW\uparrow = \alpha SW\downarrow$$

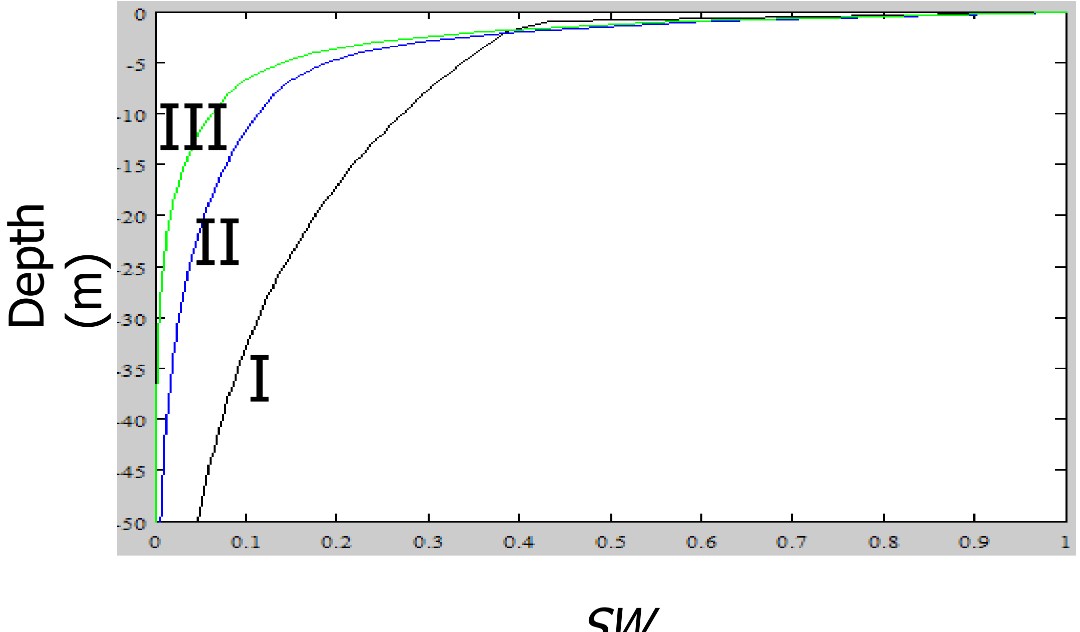
α is the albedo - typically depends on solar zenith angle and sea state

Attenuation of *SW* radiation in the water column is a function of turbidity and depth *D* (Jerlov, 1968, 1976; Paulson and Clayson, 1977):

$$SW(z) = (1-\alpha)SW\downarrow\left[\mathrm{R}e^{-D/d_1} + (1-\mathrm{R})e^{-D/d_2}\right]$$

R, d_1 and d_2 depend on water type; *D* is the distance from F.S.

Type	R	d_1 (m)	d_2 (m)
Jerlov I	0.58	0.35	23
Jerlov IA	0.62	0.60	20
Jerlov IB	0.67	1.00	17
Jerlov II	0.77	1.50	14
Jerlov III	0.78	1.40	7.9
'6' Paulson and Simpson	0.62	1.50	20
'7' Estuary	0.80	0.90	2.1



Infrared radiation

• Downwelling *IR* at the surface is forecast in NWP models - a function of air temperature, cloud cover, humidity, etc [sflux_rad*.nc]

• Upwelling *IR* is can be approximated as either a broadband measurement solely within the *IR* wavelengths (i.e., 4-50 μm), or more commonly the blackbody radiative flux

$$IR \uparrow = \varepsilon \sigma T_{sfc}^4$$

ε is the emissivity, ~1
σ is the Stefan-Boltzmann constant
T_{sfc} is the surface temperature

Turbulent Fluxes of Sensible and Latent Heat

In general, turbulent fluxes are a function of:

- T_{sfc}, T_{air}
- near-surface wind speed
- surface atmospheric pressure
- near-surface humidity

Scales of motion responsible for these heat fluxes are *much* smaller than can be resolved by any operational model - they must be parameterized (i.e., bulk aerodynamic formulation)

$$S = -\rho_a C_{pa} \overline{w'T'} = -\rho_a C_{pa} u_* T_*$$

$$E = -\rho_a L_e \overline{w'q'} = -\rho_a L_e u_* q_*$$

where:
u_* is the friction scaling velocity
T_* is the temperature scaling parameter
q_* is the specific humidity scaling parameter
ρ_a is the surface air density
C_{pa} is the specific heat of air
L_e is the latent heat of vaporization

The scaling parameters are defined using Monin-Obukhov similarity theory, and must be solved for *iteratively* (i.e., Zeng et al., 1998).

Wind Shear Stress: Turbulent Flux of Momentum

Calculation of shear stress follows naturally from calculation of turbulent heat fluxes

Total shear stress of atmosphere upon surface:

$$\tau_w = \frac{\rho_a}{\rho_0} u_*^2$$

Alternatively, Pond and Picard's formulation can be used as a simpler option

$$\boldsymbol{\tau}_w = \frac{\rho_a}{\rho_0} C_{ds} \mid \mathbf{u}_w \mid \mathbf{u}_w$$

$$C_{ds} = \frac{0.61 + 0.063 u_w^{'}}{1000}$$

$$u_w^{'} = \max(6, \min(50, u_w))$$

Atmospheric inputs to heat/momentum exchange model

- Forecasts of u_w v_w @ 10m, P_A @ MSL, T_{air} and q_{air} @2m
  - used by the bulk aerodynamic model to calculate the scaling parameters
- Forecasts of downward *IR* and *SW*

Atmospheric properties ("air")

data source	supplying agency	time period	spatial resolution	temporal resolution	area of coverage	data type
MRF	NCEP	04/01/2001-02/29/2004	1° x 1°	12-hour snapshots	129W-120W, 35N-51N	forecast
GFS	NCEP	07/03/2003-present	1° x 1°	3-hour snapshots	180W-70W, 90S-90N	forecast
OSU-ARPS	OSU	05/04/2001-02/25/2004	12 km	1-hour snapshots	128W-119W, 41N-47N (approx)	forecast
ETA/NAM	NCEP	07/03/2003-present	12 km	3-hour snapshots	Eta Grid 218, west of 100W	forecast
NCAR/NCEP Reanalysis (NARR)	NCAR	01/01/1979-12/31/2006	12km	6-hour snapshots	North America	reanalysis

Heat fluxes ("rad")

data source	supplying agency	time period	spatial resolution	temporal resolution	area of coverage	data type
AVN (lo-res)	NCEP	04/01/2001-10/28/2002	0.7° x 0.7° (approx)	3-hour averages	129W-120W, 35N-51N	forecase
AVN (hi-res)	NCEP	10/29/2002-02/29/2004	0.5° x 0.5° (approx)	3-hour averages	129W-120W, 35N-51N	forecast
GFS	NCEP	07/03/2003-present	0.5° x 0.5° (approx)	3-hour averages	180W-70W, 90S-90N	forecast
ETA/NAM	NCEP	07/03/2003-present	12 km	3-hour snapshots	Eta Grid 218, west of 100W	forecast
NCAR/NCEP Reanalysis (NARR)	NCAR	01/01/1979-12/31/2006	12km	6-hour averages	North America	reanalysis

Turbulence closure: Umlauf and Burchard (2003)

- Use a generic length-scale variable to represent various closure schemes
  - Mellor-Yamada-Galperin (with modification to wall-proximity function → *k-kl*)
  - *k-ε* (Rodi)
  - *k-ω* (Wilcox)
  - KPP (Large et al.; Durski et al)

$$\begin{cases} \frac{Dk}{Dt} = \frac{\partial}{\partial z}\left(\nu_k^{\psi}\frac{\partial k}{\partial z}\right) + \nu M^2 + \kappa N^2 - \epsilon + c_{fk}\alpha|\mathbf{u}|^3\mathcal{H}(z_v - z) \\ \frac{D\psi}{Dt} = \frac{\partial}{\partial z}\left(\nu_{\psi}\frac{\partial \psi}{\partial z}\right) + \frac{\psi}{k}\left[c_{\psi1}\nu M^2 + c_{\psi3}\kappa N^2 - c_{\psi2}\epsilon F_{wall} + c_{f\psi}\alpha|\mathbf{u}|^3\mathcal{H}(z_v - z)\right] \end{cases}$$

$$\psi = \left(c_\mu^0\right)^p k^m \ell^n$$

$$\varepsilon = \left(c_\mu^0\right)^3 k^{3/2}\ell^{-1}$$

Wall function

$$F_{wall} = 1 + E_2\left(\frac{l}{\kappa_0 d_b}\right)^2 + E_4\left(\frac{l}{\kappa_0 d_s}\right)^2$$

Diffusivity

$$\nu = c_\mu k^{1/2}\ell \qquad \kappa = c_\mu' k^{1/2}\ell \qquad \nu_k^{\psi} = \frac{\nu}{\sigma_k^{\psi}} \qquad \nu_{\psi} = \frac{\nu}{\sigma_{\psi}}$$

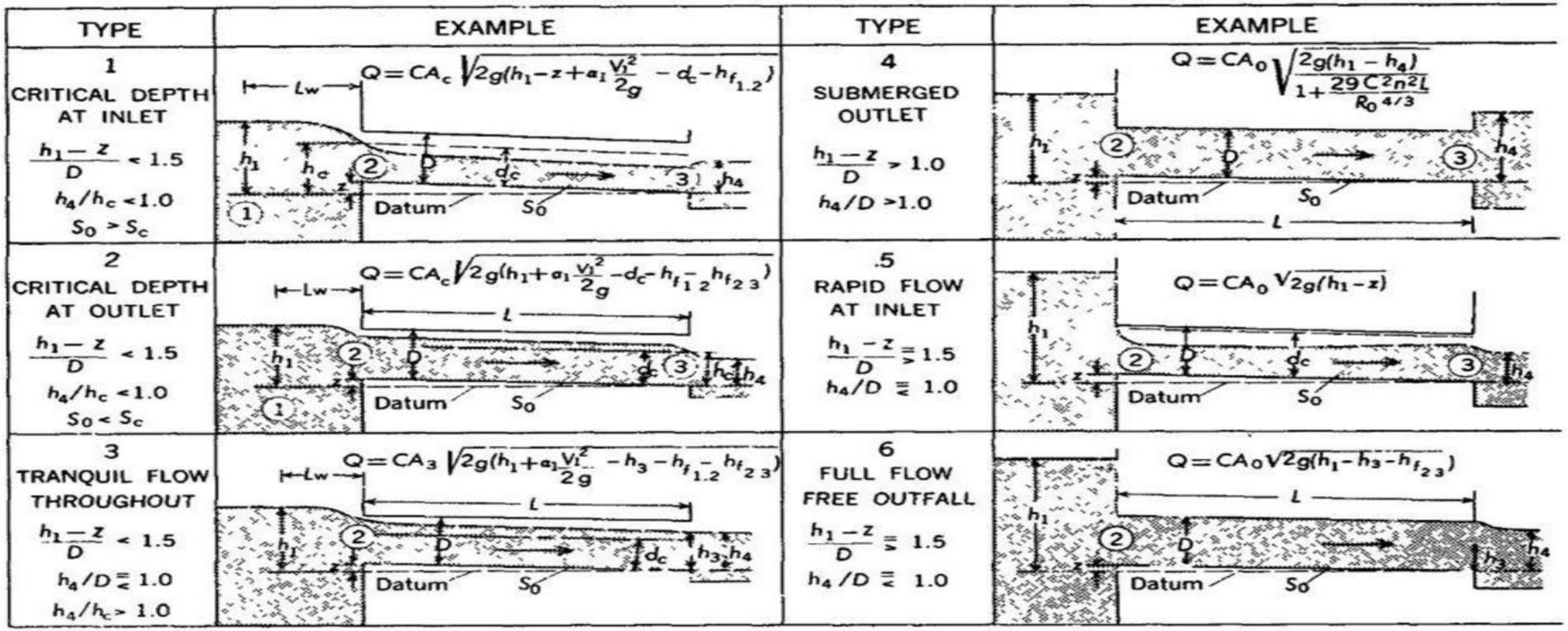
Stability: Kantha and Clayson (smoothes Galperin’s stability function as it approaches max)

$$c_\mu = \sqrt{2}s_m, \quad c_\mu' = \sqrt{2}s_h \qquad s_m = \frac{0.392 + 17.07 s_h G_h}{1 - 6.127 G_h}$$

$$s_h = \frac{0.4939}{1 - 30.19 G_h} \qquad -0.28 \le G_h = \frac{G_{h\_u} - (G_{h\_u} - G_{h\_c})^2}{G_{h\_u} + G_{h0} - 2G_{h\_c}} \le 0.023$$

$$G_{h\_c} = 0.02$$

Classification of six types of culvert flow



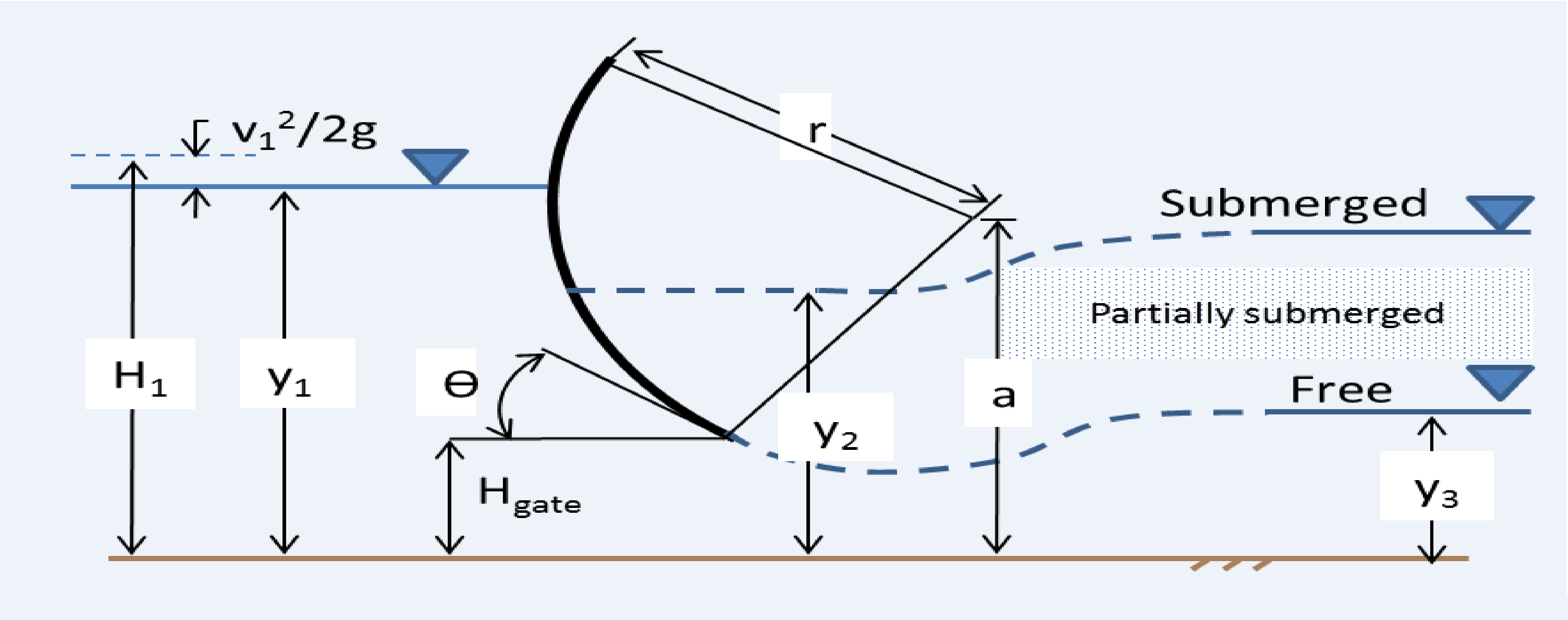
TYPE	EXAMPLE	TYPE	EXAMPLE
1 CRITICAL DEPTH AT INLET $\frac{h_1 - z}{D} < 1.5$ $h_4/h_c < 1.0$ $S_0 > S_c$	$Q = CA_c\sqrt{2g\left(h_1 - z + \alpha_1\frac{V_1^2}{2g} - d_c - h_{f_{1,2}}\right)}$	4 SUBMERGED OUTLET $\frac{h_1 - z}{D} > 1.0$ $h_4/D > 1.0$	$Q = CA_0\sqrt{\frac{2g(h_1 - h_4)}{1 + \frac{29C^2n^2L}{R_0^{4/3}}}}$
2 CRITICAL DEPTH AT OUTLET $\frac{h_1 - z}{D} < 1.5$ $h_4/h_c < 1.0$ $S_0 < S_c$	$Q = CA_c\sqrt{2g\left(h_1 + \alpha_1\frac{V_1^2}{2g} - d_c - h_{f_{1,2}} - h_{f_{2,3}}\right)}$	5 RAPID FLOW AT INLET $\frac{h_1 - z}{D} \geq 1.5$ $h_4/D \leq 1.0$	$Q = CA_0\sqrt{2g(h_1 - z)}$
3 TRANQUIL FLOW THROUGHOUT $\frac{h_1 - z}{D} < 1.5$ $h_4/D \leq 1.0$ $h_4/h_c > 1.0$	$Q = CA_3\sqrt{2g\left(h_1 + \alpha_1\frac{V_1^2}{2g} - h_3 - h_{f_{1,2}} - h_{f_{2,3}}\right)}$	6 FULL FLOW FREE OUTFALL $\frac{h_1 - z}{D} \geq 1.5$ $h_4/D \leq 1.0$	$Q = CA_0\sqrt{2g(h_1 - h_3 - h_{f_{2,3}})}$

Orifice equation

$$Q = CA\sqrt{2g(y_1 - y_3)}$$

where A = Area of the gate opening
C = Discharge coefficient (typically 0.8)

Hydraulics



Radial gate

weir