SCHISM numerical formulation

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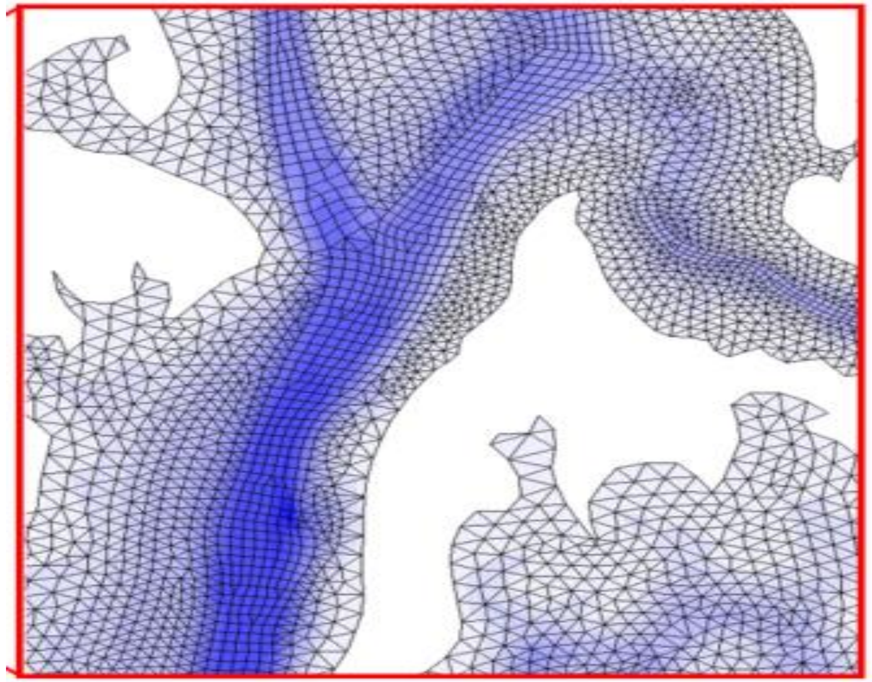
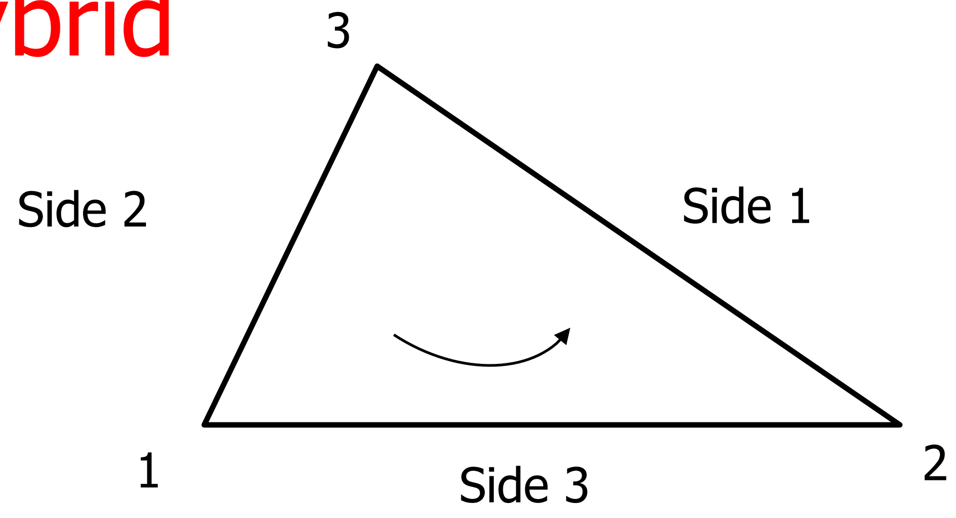
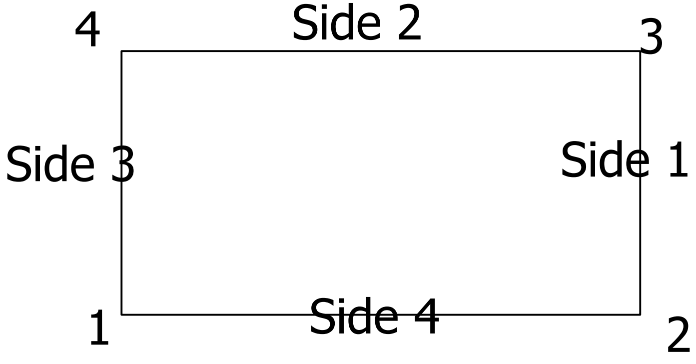
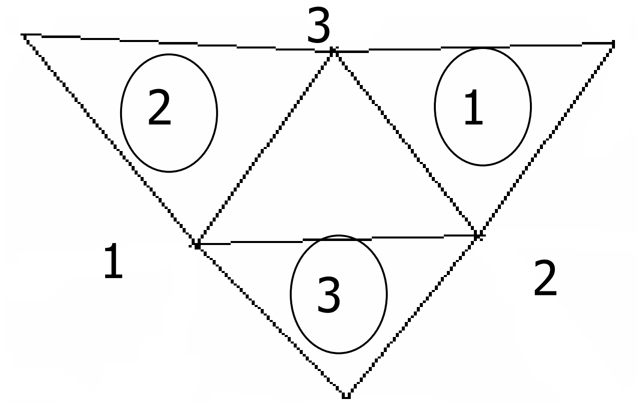
Horizontal grid: hybrid

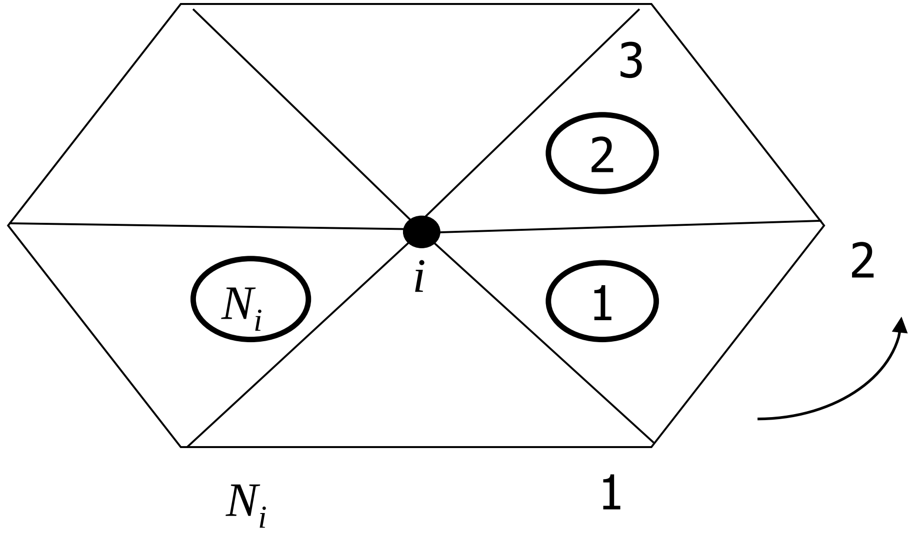


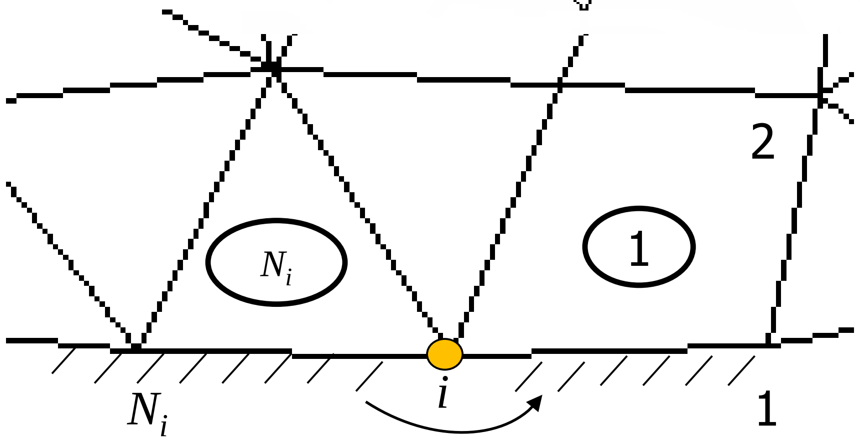


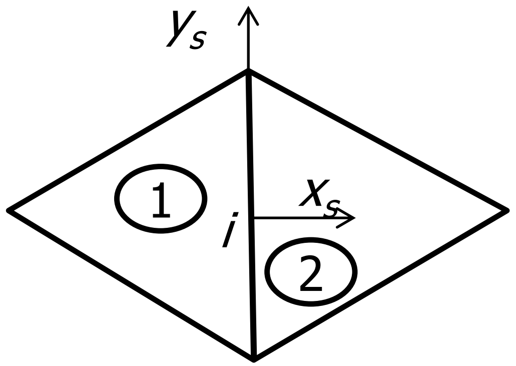




Counter-clockwise convention

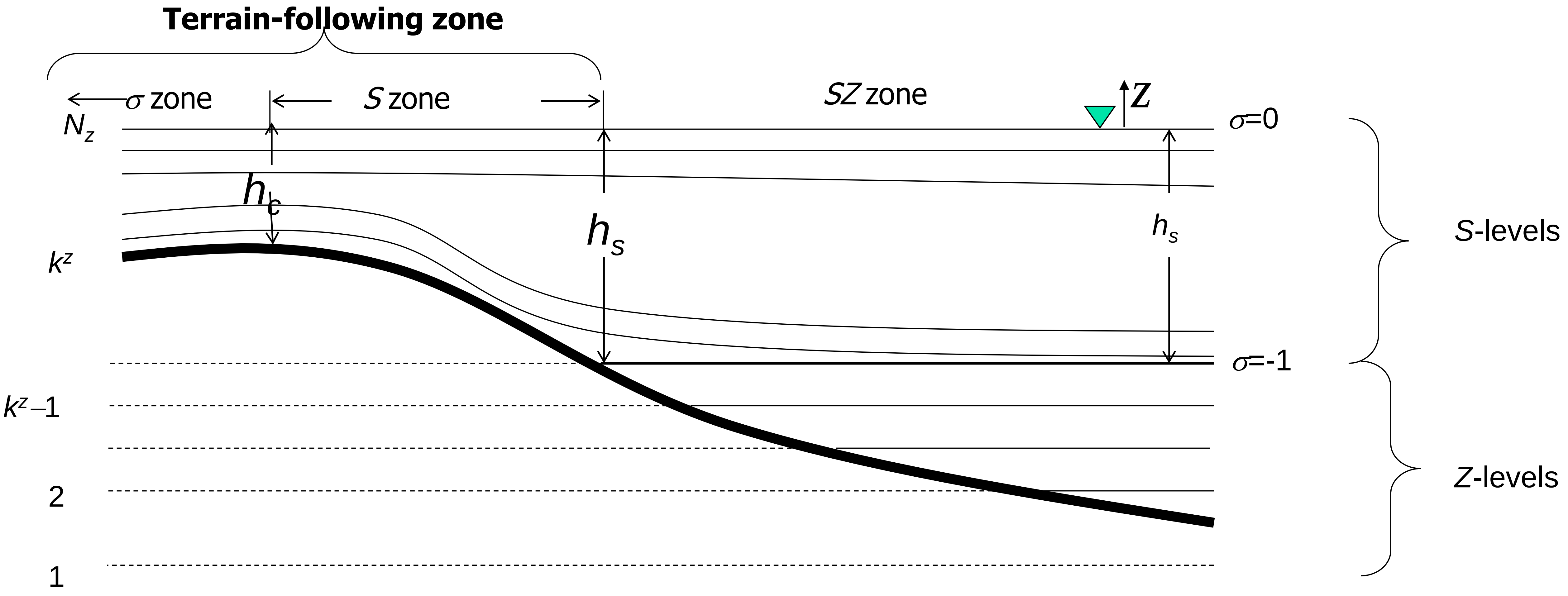






- Internal ‘ball’: starting elem is arbitrary
- Boundary ball: starting elem is unique

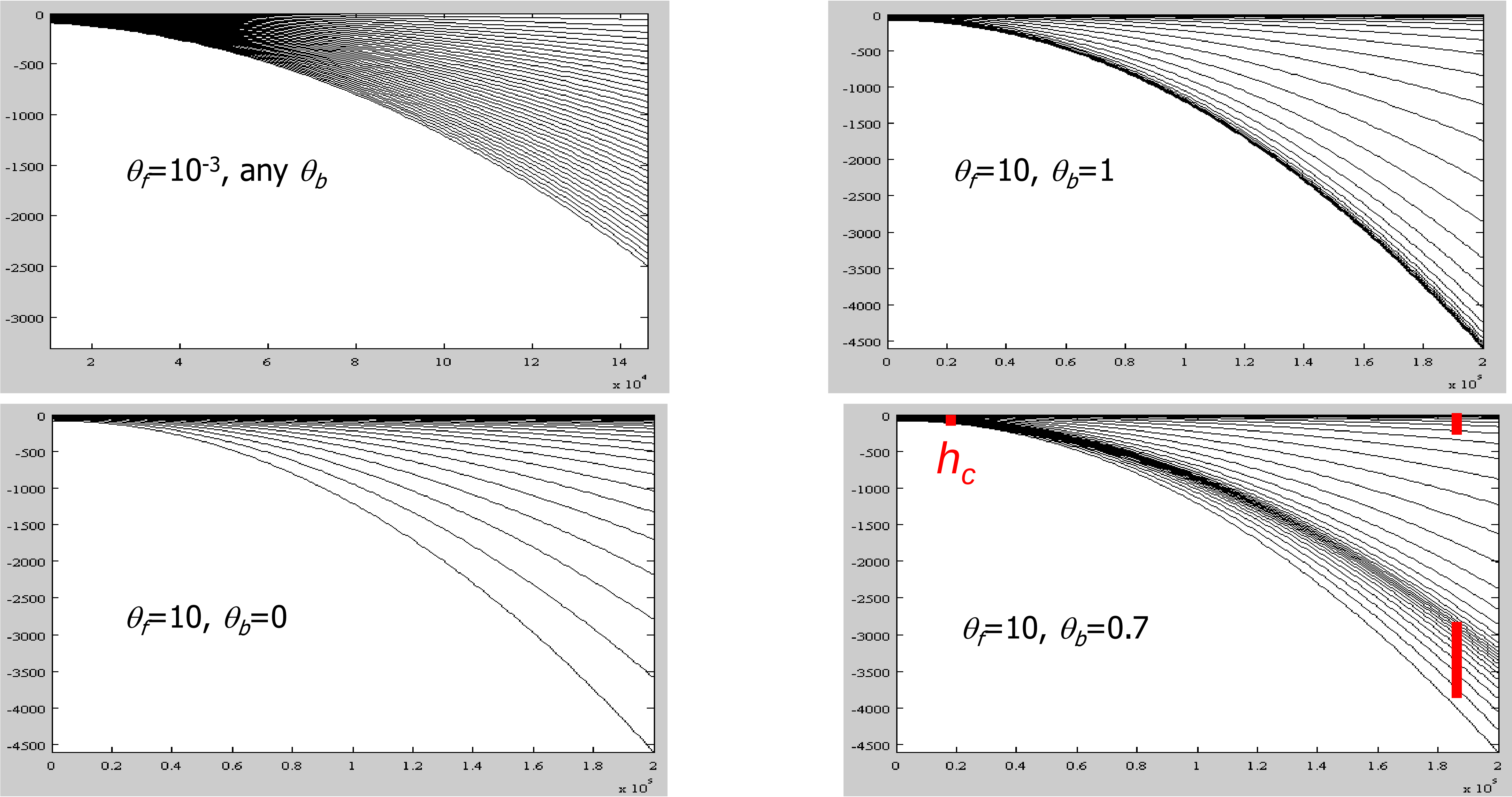
Vertical grid (1): *SZ*



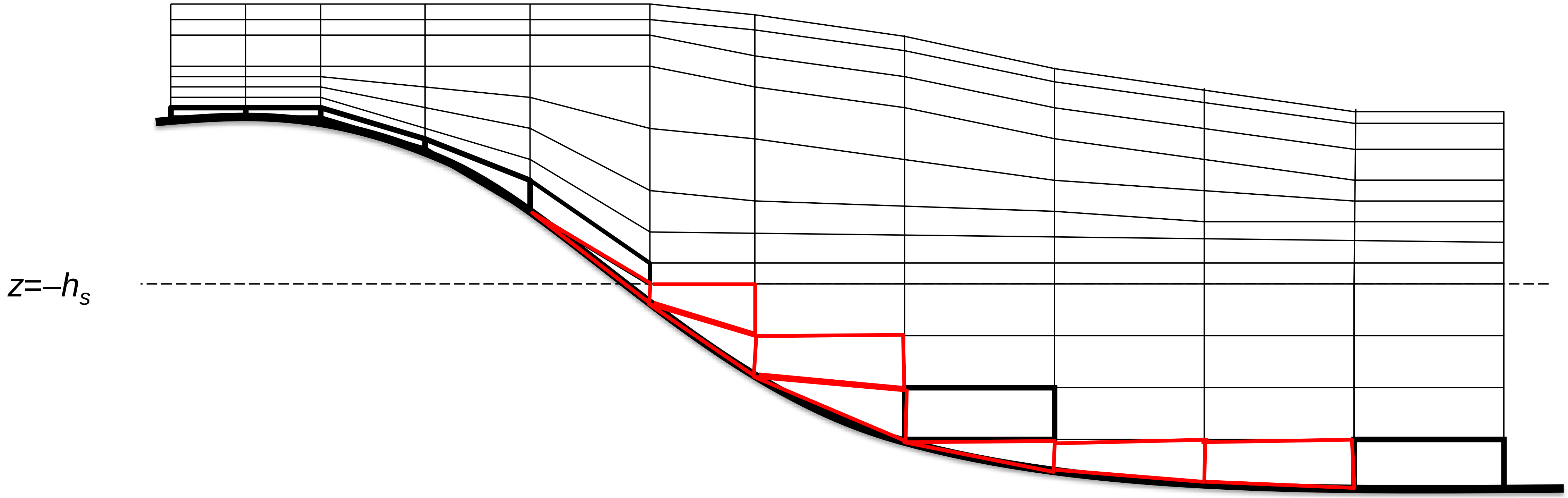
$$\begin{cases} z = \eta(1+\sigma) + h_c\sigma + (\tilde{h} - h_c)C(\sigma) & (-1 \le \sigma \le 0) \\ C(\sigma) = (1-\theta_b)\dfrac{\sinh(\theta_f\sigma)}{\sinh\theta_f} + \theta_b\dfrac{\tanh\left[\theta_f(\sigma+1/2)\right] - \tanh(\theta_f/2)}{2\tanh(\theta_f/2)} & (0 \le \theta_b \le 1;\quad 0 < \theta_f \le 20) \end{cases}$$

$$\tilde{h} = \min(h, h_s)$$

S-coordinates

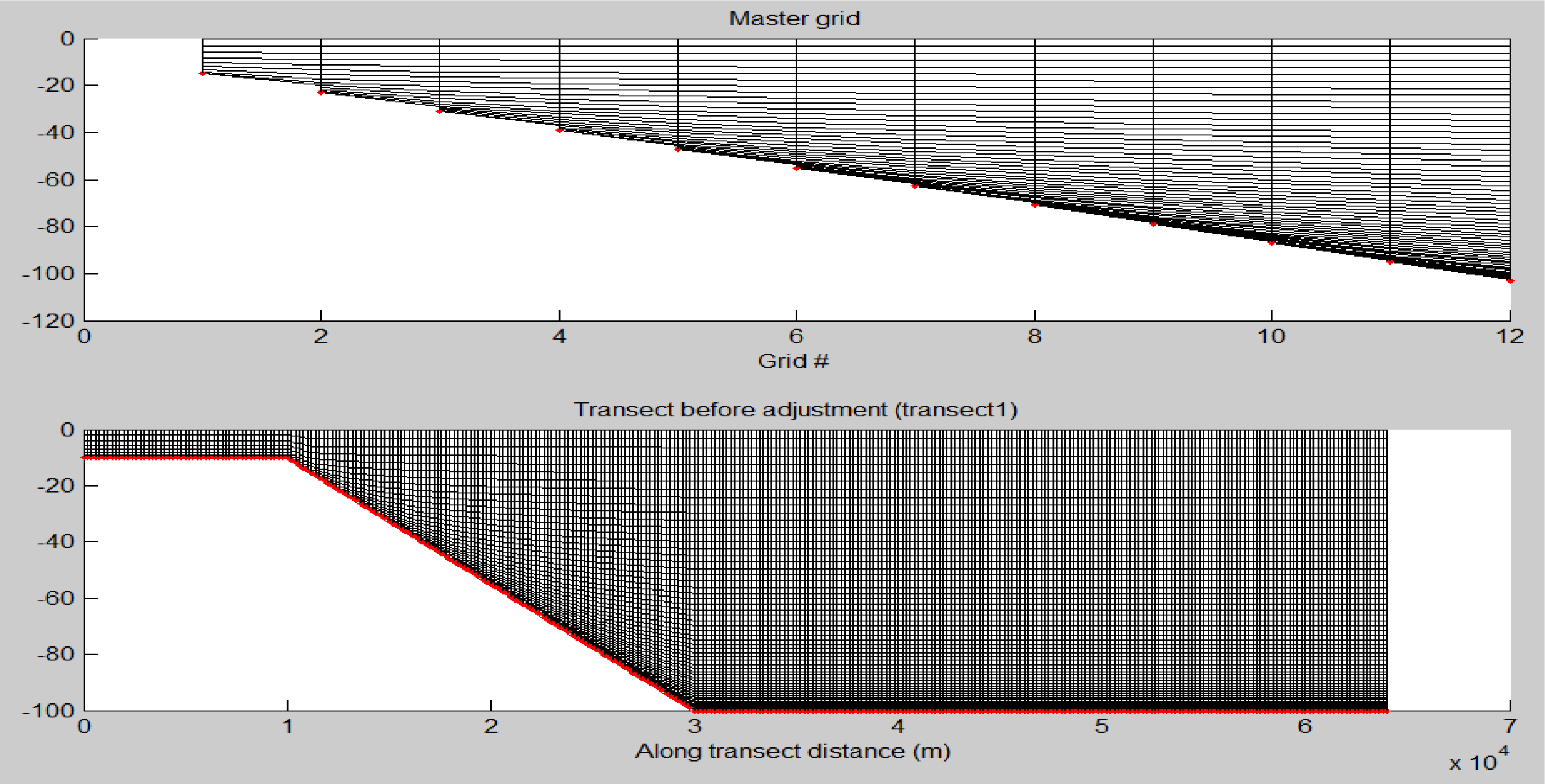


Vertical grid: SZ

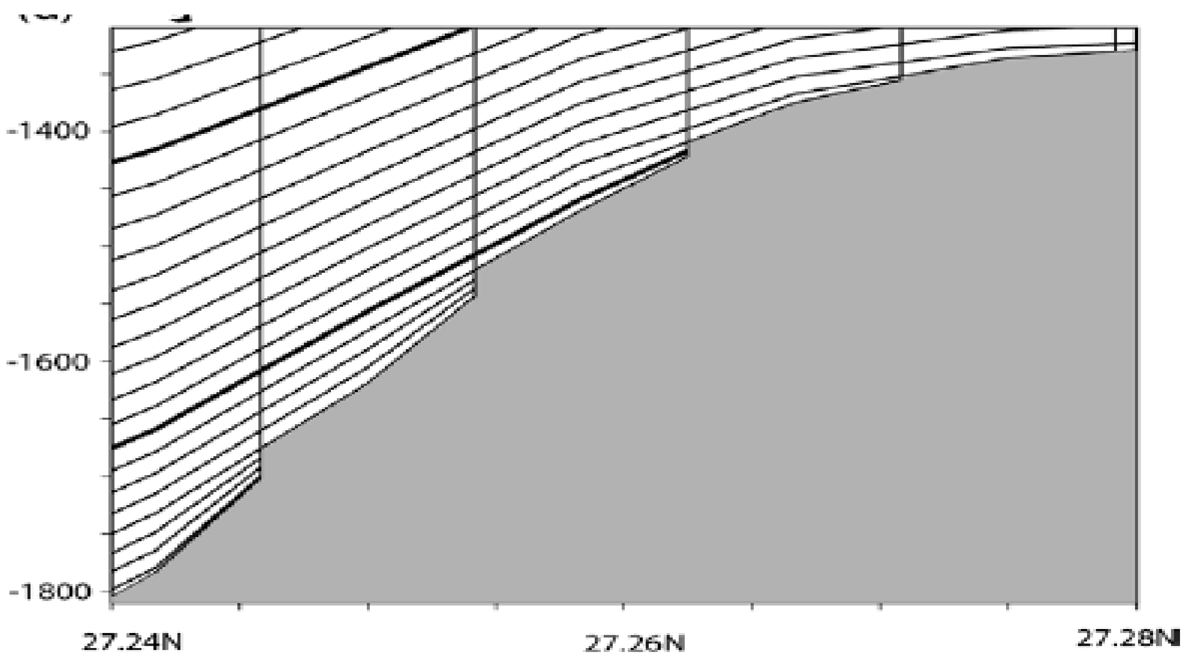


The bottom cells are shaved – no staircases!

Vertical grid (2): LSC² via VQS



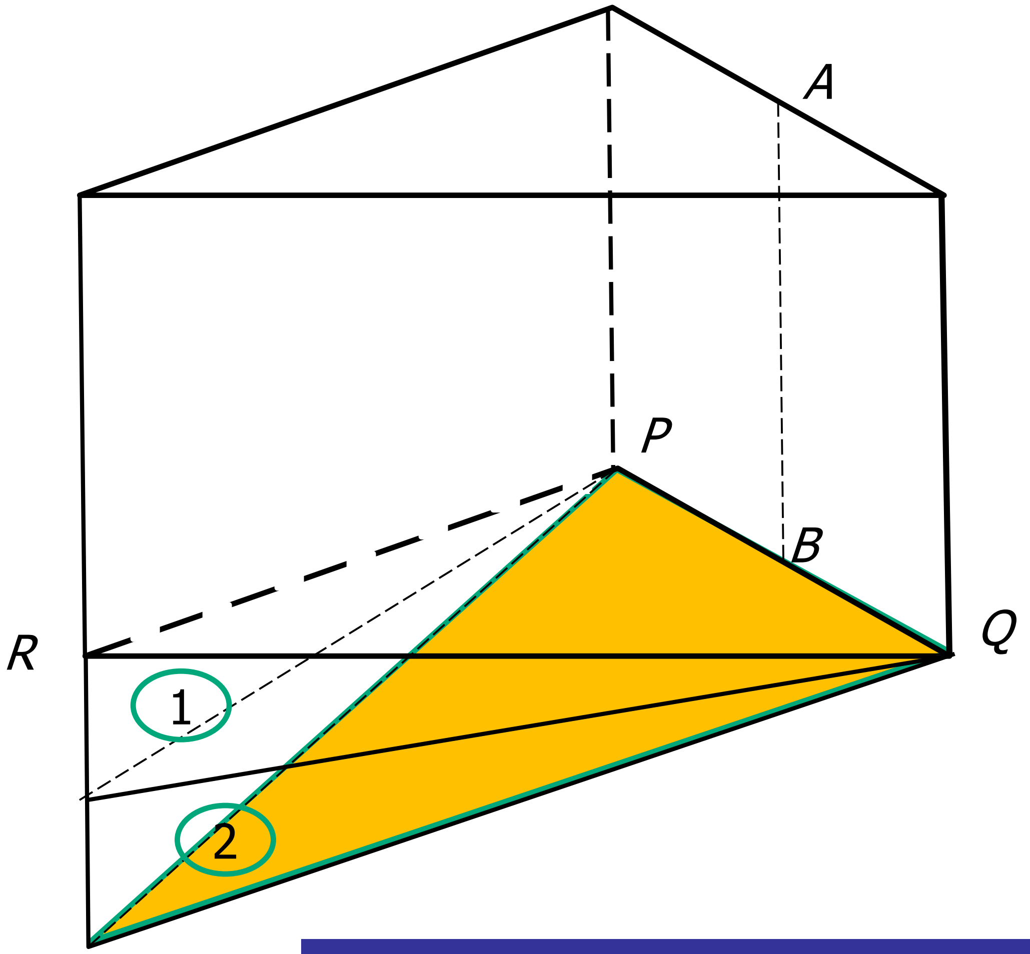
Dukhovskoy et al. (2009)

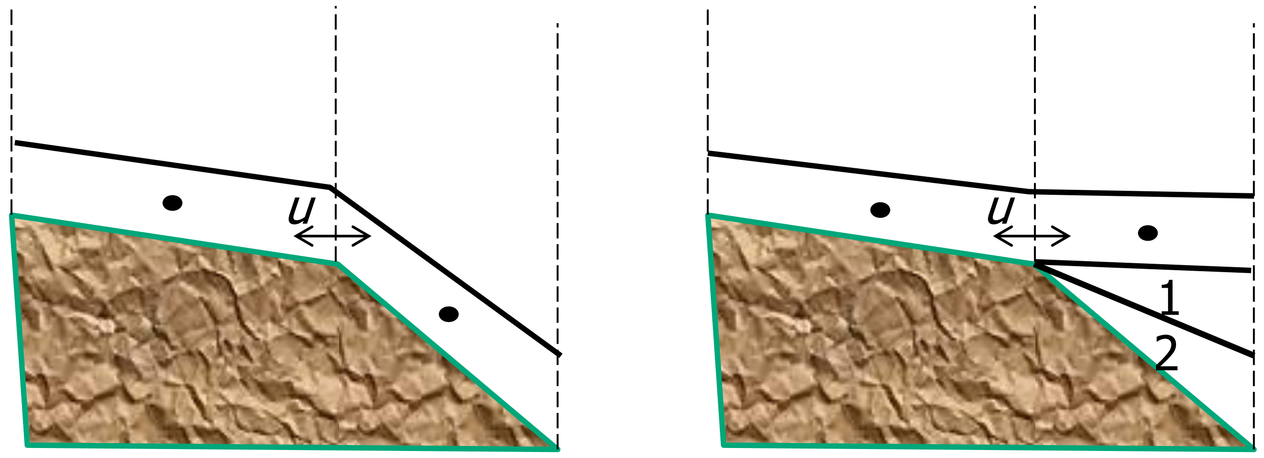


Procedure for VQS

- Different # of levels are used at different depths
- At a given depth, the # of levels is interpolated from the 2 adjacent master grids
- Thin layers near the bottom are masked
- Staircases appear near the bottom (like *Z*), but otherwise terrain following

What to do with the staircases?



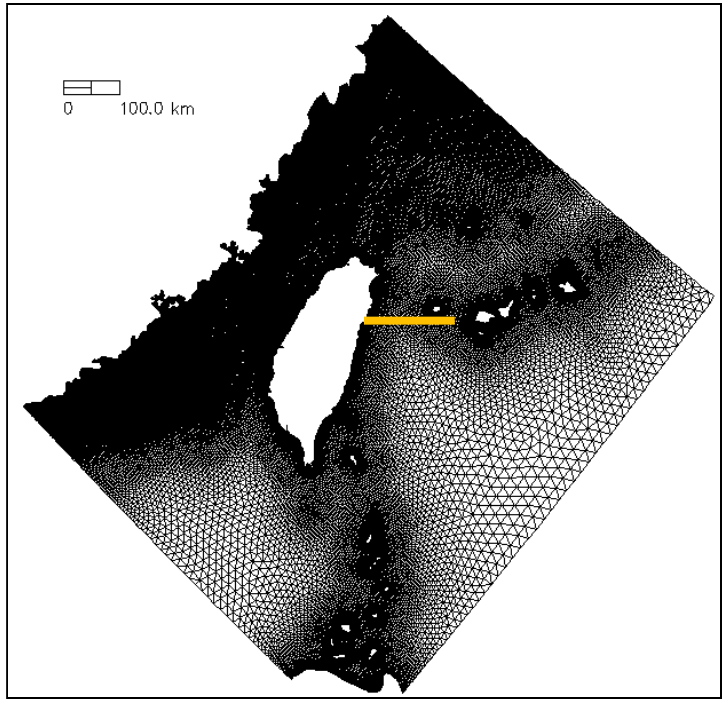


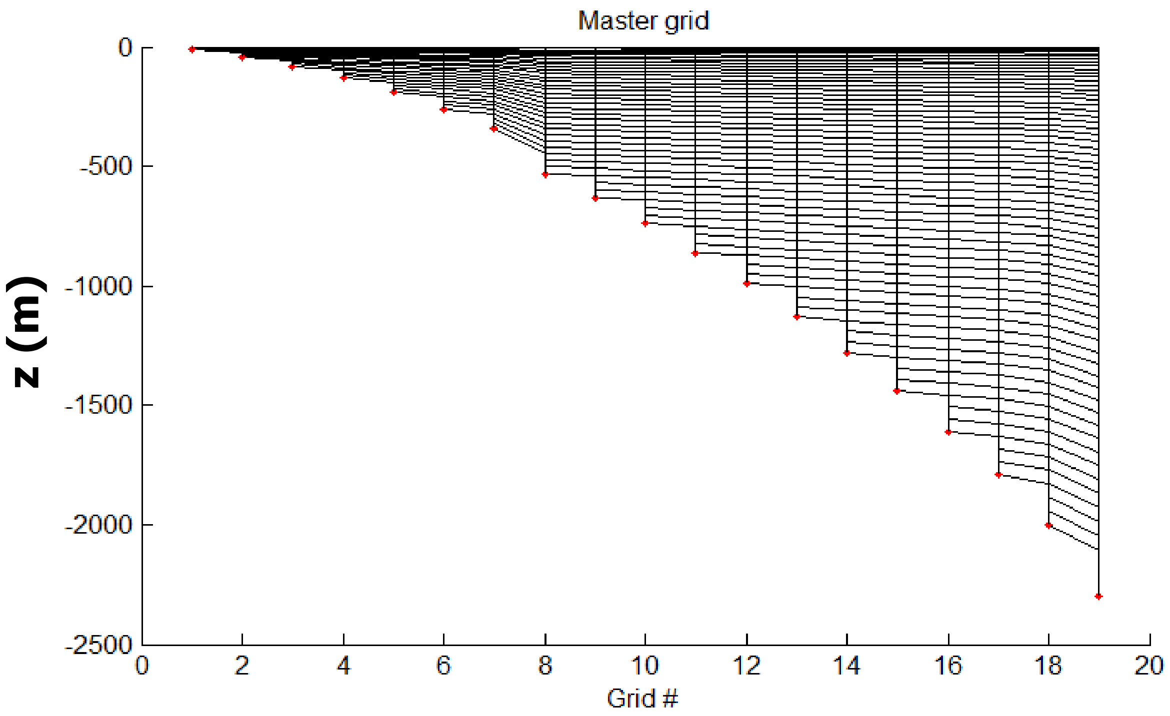
LSC²= Localized Sigma Coordinates with Shaved Cells

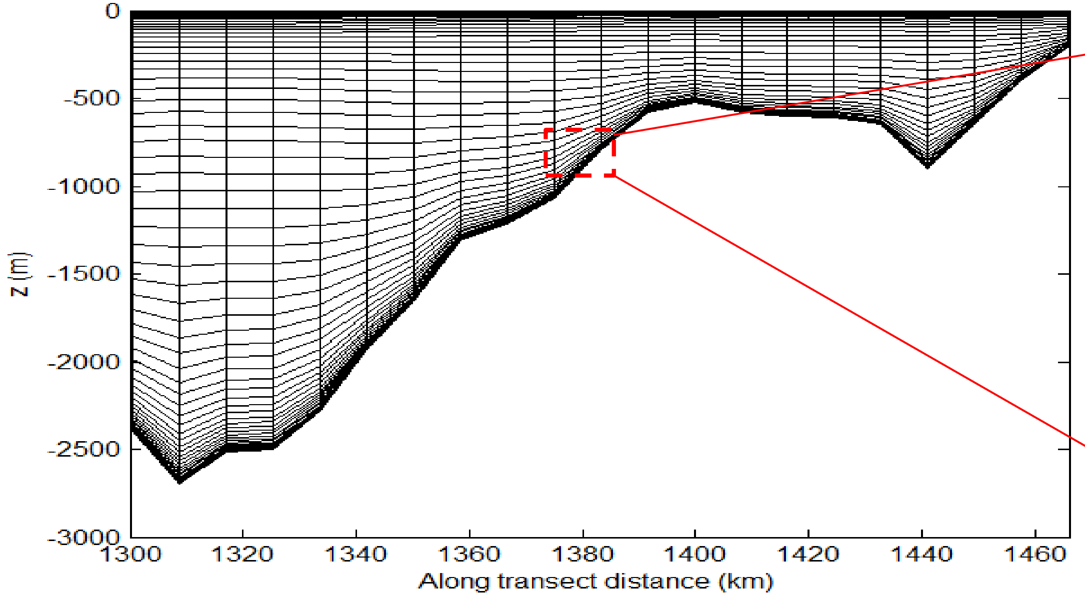
(*ivcor* =1 in vgrid.in)

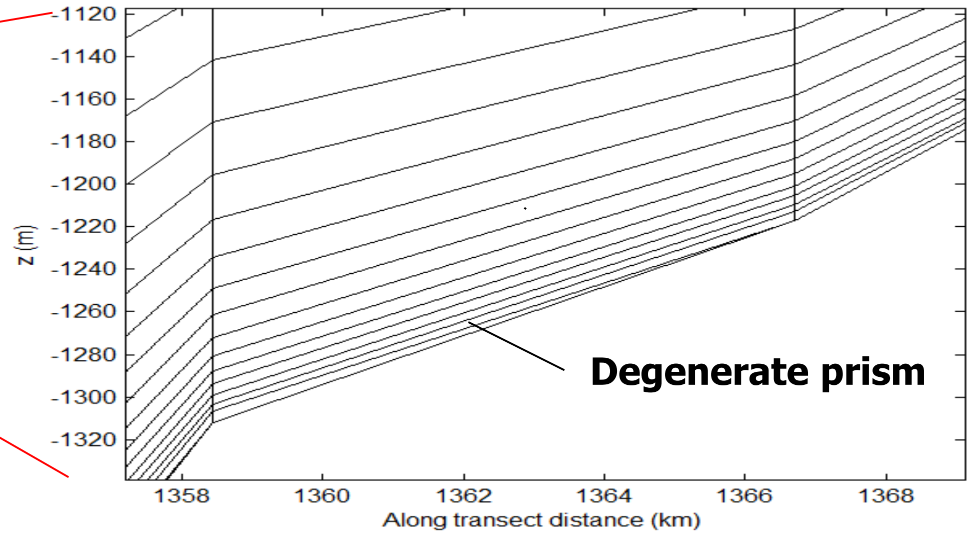
*No masking of thin layers (c/o implicit scheme)

Vertical grid: LSC^2



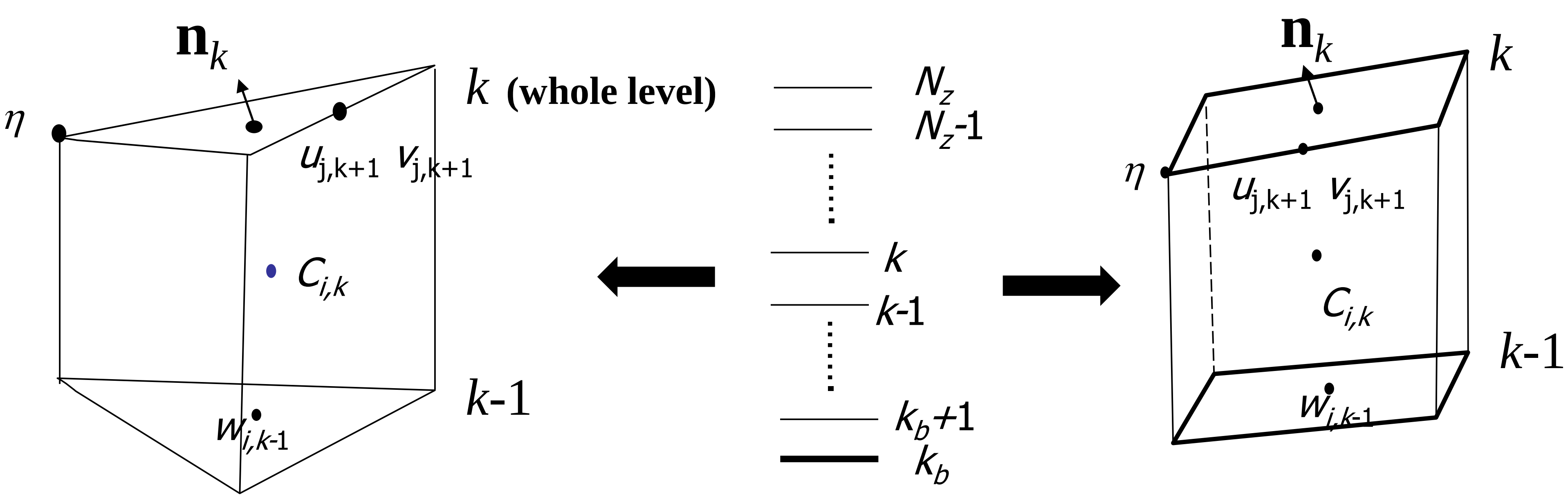






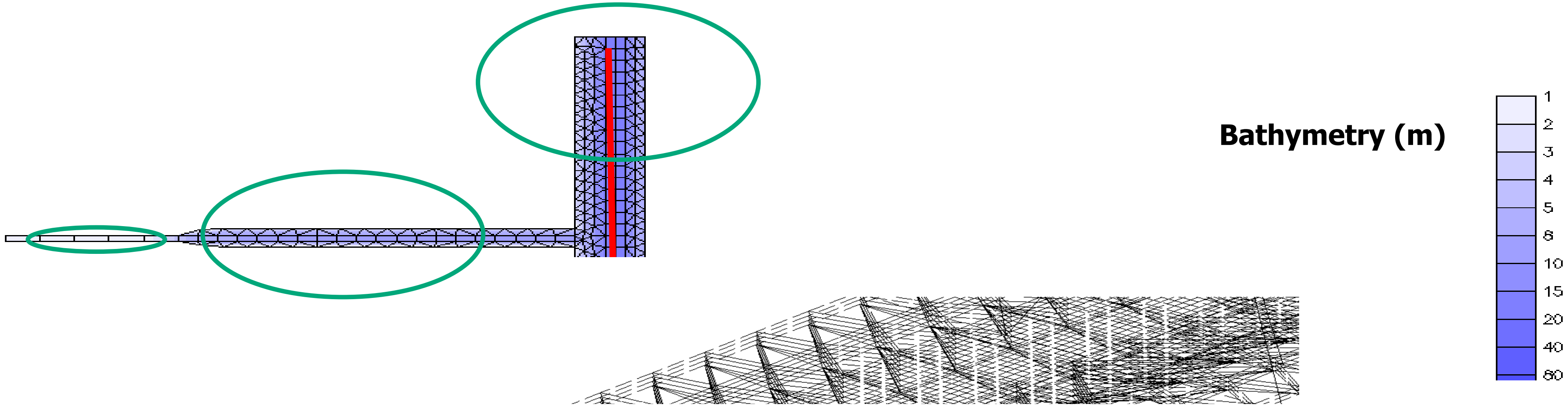
Vertical grid (3)

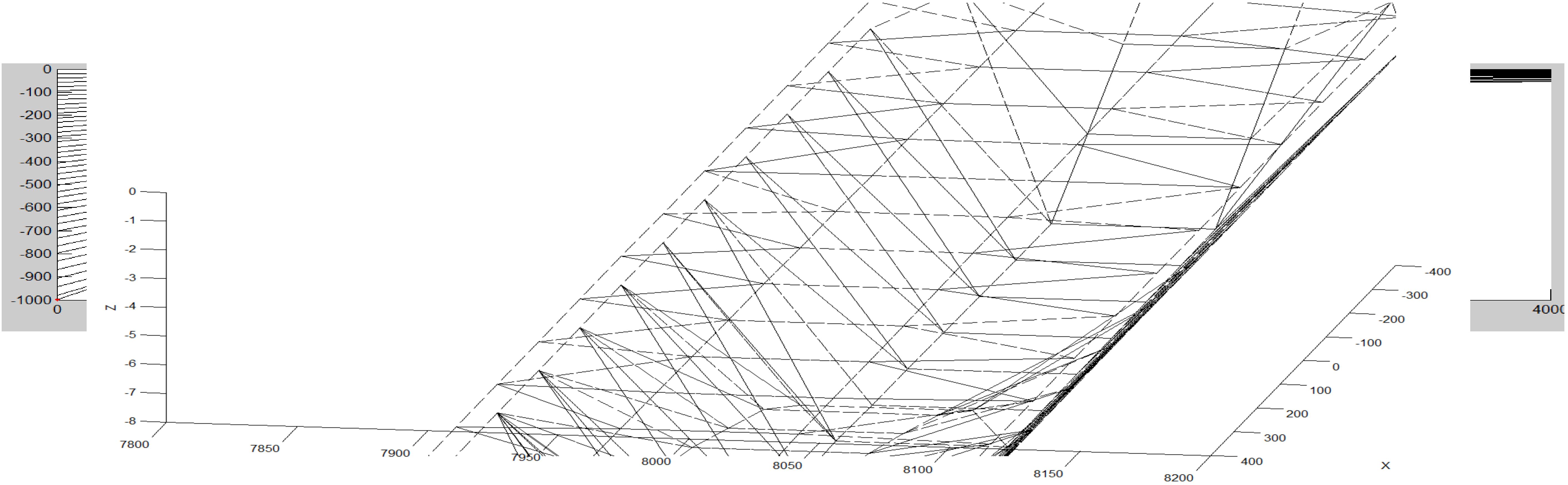
- 3D computational unit: uneven prisms
- **Equations are not transformed into *S*- or σ-coordinates, but solved in the original *Z*-space**
- Pressure gradient $\int_z^\eta \nabla \rho d\zeta$
  - •*Z*-method with cubic spline (extrapolation on surface/bottom)



Polymorphism

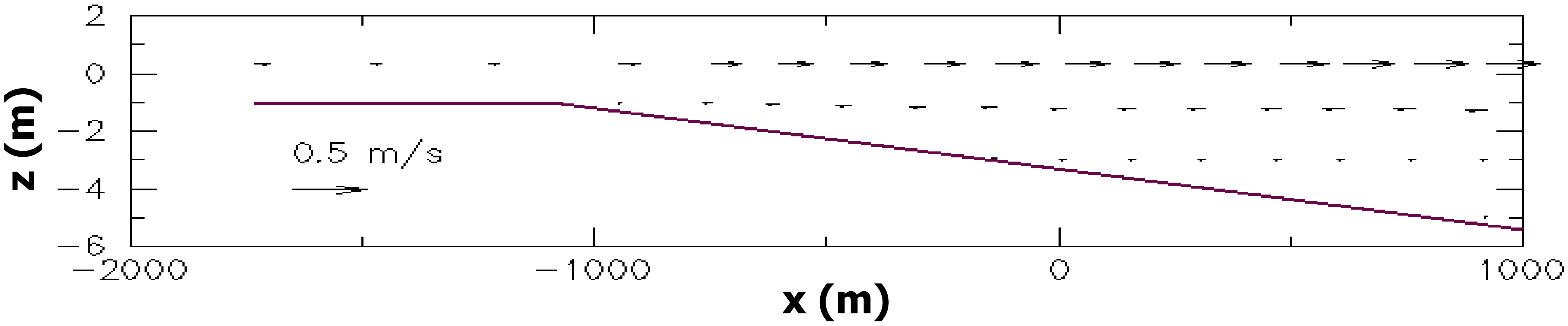
Zhang et al. (2016)

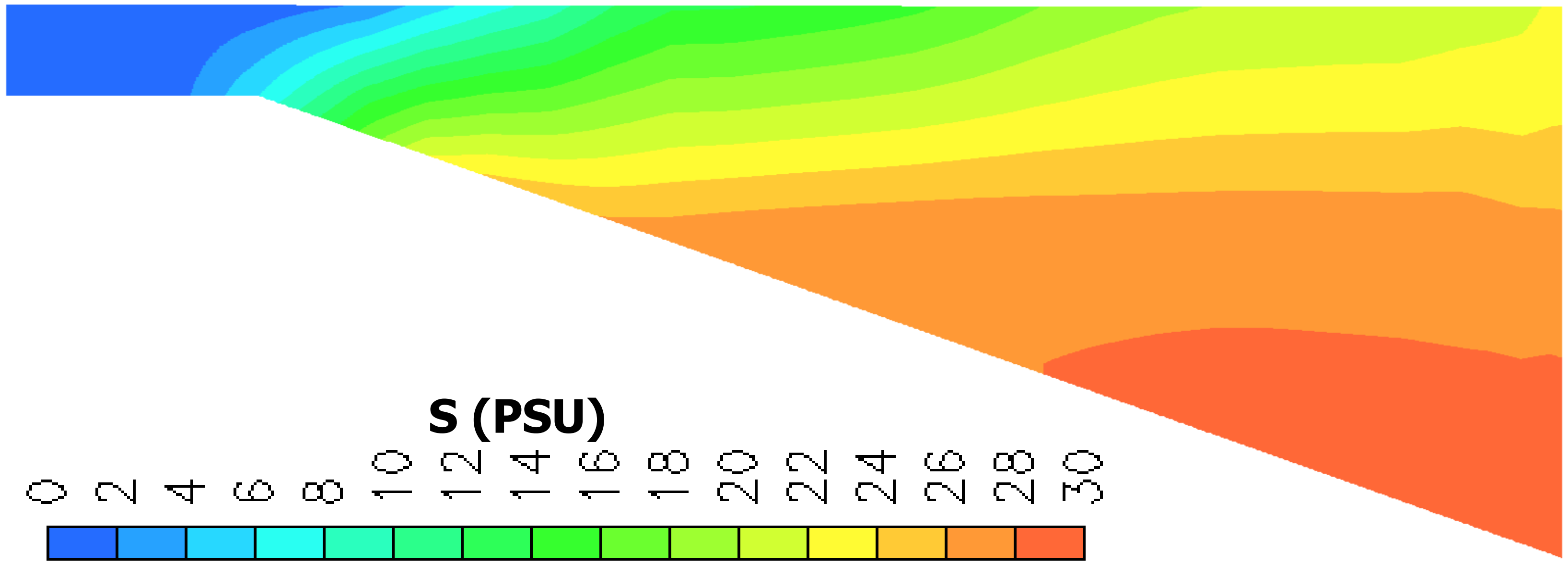




Polymorphism

flood





*Make sure there is no jump in Cd between 2D/3D zones

Semi-implicit scheme

Continuity

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t}+\theta\nabla\cdot\int_{-h}^{\eta}\mathbf{u}^{n+1}dz+(1-\theta)\nabla\cdot\int_{-h}^{\eta}\mathbf{u}^{n}dz=0$$

Momentum

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{*}}{\Delta t}=\boldsymbol{f}-g\theta\nabla\eta^{n+1}-g(1-\theta)\nabla\eta^{n}+\boldsymbol{m}_{\boldsymbol{z}}^{\boldsymbol{n+1}}-\alpha|\boldsymbol{u}|\boldsymbol{u}^{\boldsymbol{n+1}}L(x,y,z)$$

b.c. (3D)

$$\begin{cases}\nu^{n}\dfrac{\partial\mathbf{u}^{n+1}}{\partial z}=\boldsymbol{\tau}_{w}^{n+1}, & \text{at } z=\eta^{n};\\ \nu^{n}\dfrac{\partial\mathbf{u}^{n+1}}{\partial z}=\chi^{n}\mathbf{u}_{b}^{n+1}, & \text{at } z=-h,\end{cases}\qquad \chi^{n}=C_{D}\,|\,\mathbf{u}_{b}^{n}\,|$$

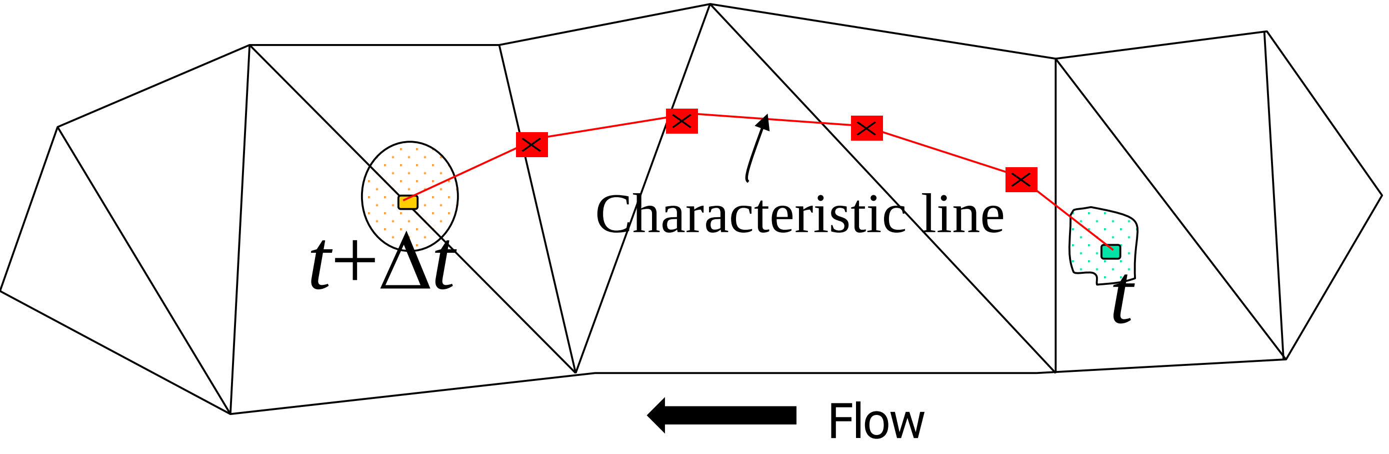
Advection:

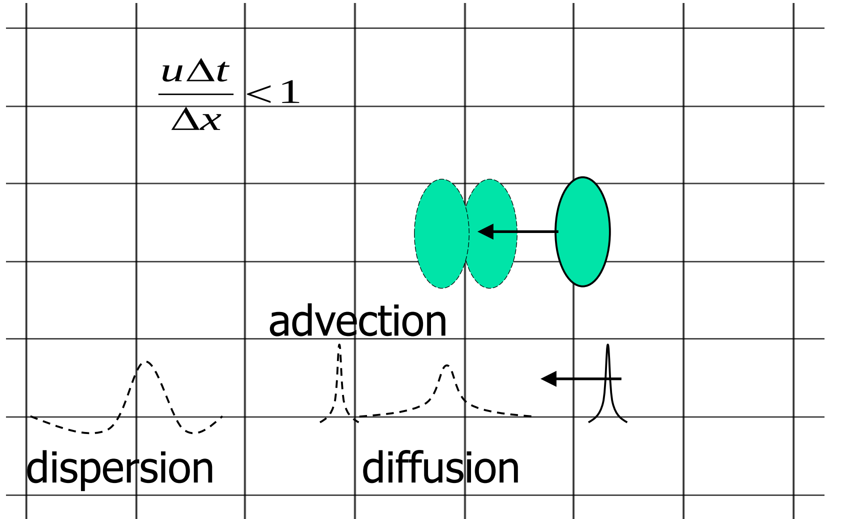
$$\frac{D\mathbf{u}}{Dt}\approx\frac{\mathbf{u}^{n+1}-\mathbf{u}_{*}}{\Delta t};\quad \mathbf{u}_{*}(x,y,z,t+\Delta t,\Delta t)$$

- Implicit treatment of divergence, pressure gradient and SAV form drag terms by-passes most severe stability constraints: $0.5\le\theta\le1$
- Explicit treatment: Coriolis, baroclinicity, horizontal viscosity, radiation stress…

Eulerian-Lagrangian Method (ELM)

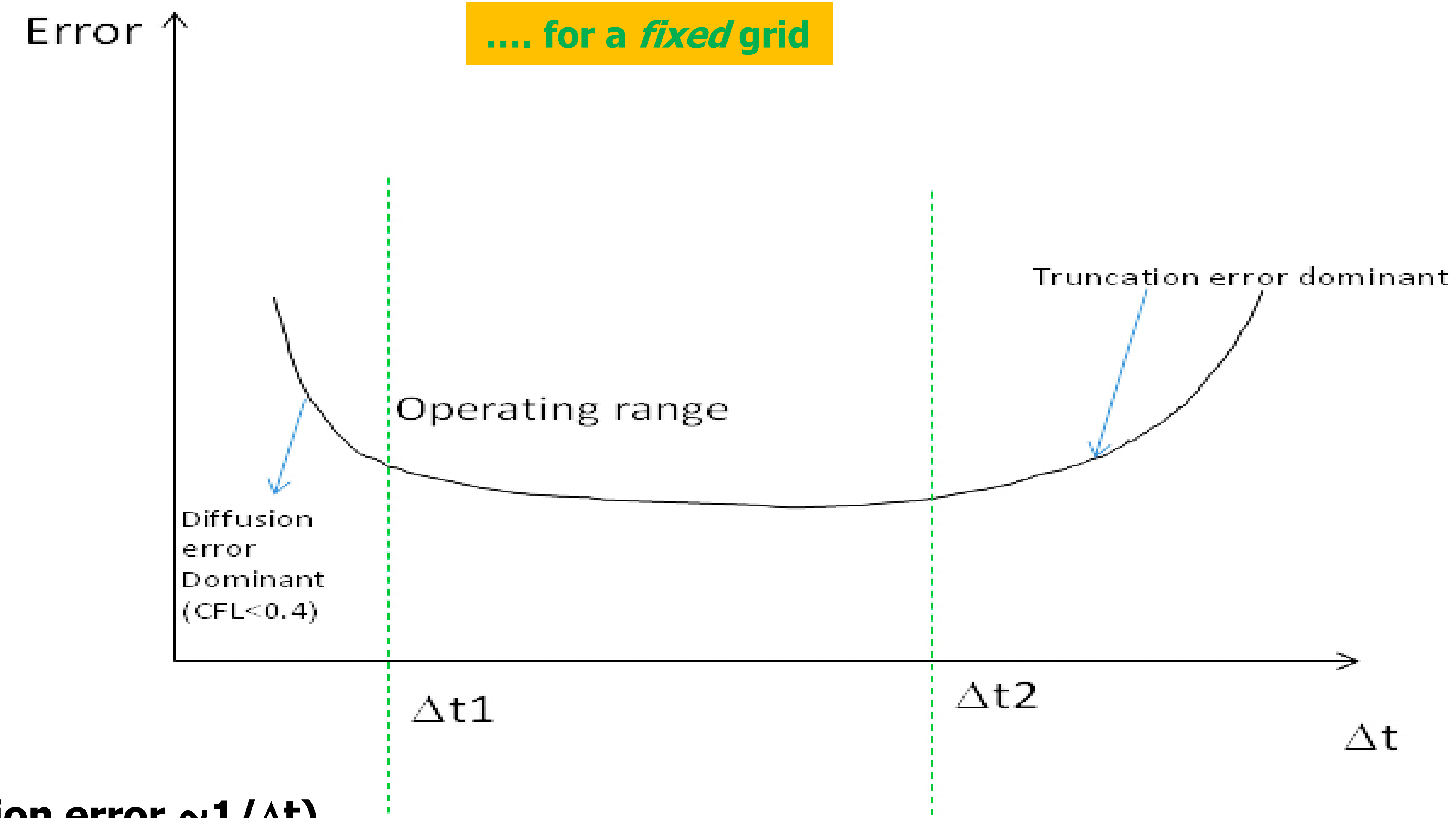
- ELM: takes advantage of both Lagrangian and Eulerian methods
  - Grid is fixed in time, and time step is *not* limited by Courant number condition
  - Advections are evaluated by following a particle that starts at certain point at time t and ends right at a pre-given point at time $t+\Delta t$.
  - The process of finding the starting point of the path (foot of characteristic line) is called ***backtracking***, which is done by integrating $d\mathbf{x}/dt=\mathbf{u}_3$ backward in 3D space and time.
  - To better capture the particle movement, the backward integration is often carried out in small sub-time steps
    - Simple backward Euler method
    - 2nd-order R-K method
    - Interpolation-ELM does not conserve; integration ELM does
  - Interpolation: numerical diffusion vs. dispersion





Operating range of SCHISM

$$CFL = \frac{(|\boldsymbol{u}| + \sqrt{gh})\Delta t}{\Delta x}$$



- **ELM prefers larger Δt (truncation error ~1/Δt)**
  - **As a result, SCHISM requires CFL>0.4**
- **Convergence: CFL=const, Δt→0**
- **For barotropic field applications: [100,450] sec**
- **For baroclinic field applications: [100,200] sec (e.g., start with 150 sec)**

ELM with high-order Kriging

- Best linear unbiased estimator for a random function $f(\mathbf{x})$
- "Exact" interpolator
- Works well on unstructured grid (efficient)
- Needs filter (ELAD) to reduce dispersion

$$f(x,y) = \underbrace{\alpha_1 + \alpha_2 x + \alpha_3 y}_{\text{Drift function}} + \underbrace{\sum_{i=1}^{N} \beta_i K(|\,\mathbf{x} - \mathbf{x}_i\,|)}_{\text{Fluctuation}}$$

K is called generalized covariance function

Minimizing the variance of the fluctuation we get

Cubic spline ↓

$$K(h) = -h, \quad h^2 \ln h, \quad h^3, \ldots$$

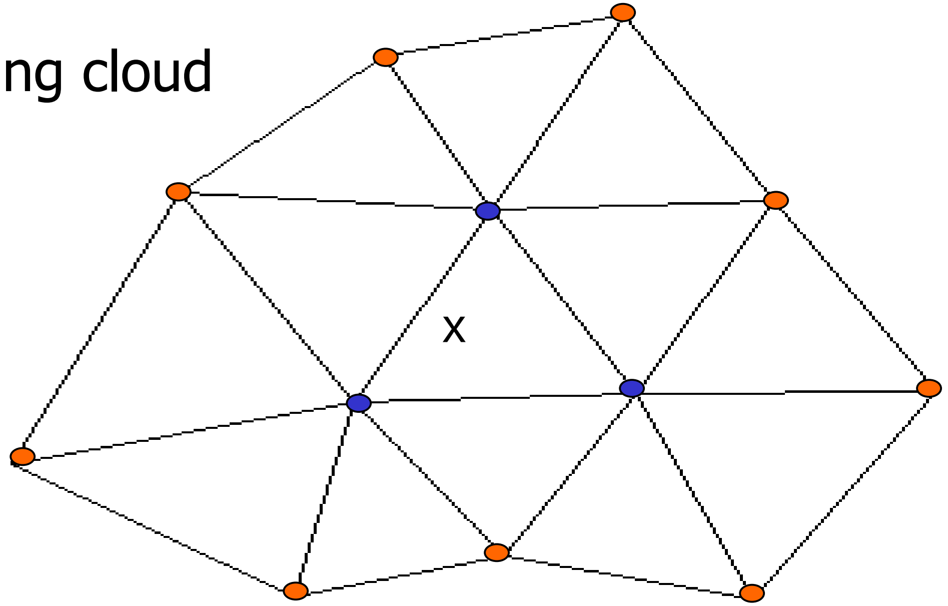
$$\sum_{i=1}^{N} \beta_i = 0$$

$$\sum_{i=1}^{N} \beta_i x_i = 0$$

$$\sum_{i=1}^{N} \beta_i y_i = 0$$

$$f(x_i, y_i) = f_i$$

2-tier Kriging cloud



- The numerical dispersion generated by kriging is reduced by a diffusion ELAD filter (Zhang et al. 2016)

Semi-implicit scheme

Continuity

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t}+\theta\nabla\cdot\int_{-h}^{\eta}\mathbf{u}^{n+1}dz+(1-\theta)\nabla\cdot\int_{-h}^{\eta}\mathbf{u}^{n}dz=0$$

Momentum

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{*}}{\Delta t}=\boldsymbol{f}-g\theta\nabla\eta^{n+1}-g(1-\theta)\nabla\eta^{n}+\boldsymbol{m}_{\boldsymbol{z}}^{\boldsymbol{n+1}}-\alpha|\boldsymbol{u}|\boldsymbol{u}^{\boldsymbol{n+1}}L(x,y,z)$$

b.c.

$$\begin{cases} \nu^{n}\dfrac{\partial\mathbf{u}^{n+1}}{\partial z}=\boldsymbol{\tau}_{w}^{n+1}, \quad \text{at} \ \ z=\eta^{n}; \\ \nu^{n}\dfrac{\partial\mathbf{u}^{n+1}}{\partial z}=\chi^{n}\mathbf{u}_{b}^{n+1}, \quad \text{at} \ \ z=-h, \end{cases} \qquad \chi^{n}=C_{D}\,|\,\mathbf{u}_{b}^{n}\,|$$

$$\frac{D\mathbf{u}}{Dt}\approx\frac{\mathbf{u}^{n+1}-\mathbf{u}_{*}}{\Delta t}; \quad \mathbf{u}_{*}(x,y,z,t+\Delta t,\Delta t)$$

- Implicit treatment of divergence and pressure gradient terms by-passes most severe CFL condition : $0.5 \leq \theta \leq 1$
- Explicit treatment: Coriolis, baroclinicity, horizontal viscosity, radiation stress…

Galerkin finite-element formulation

A Galerkin weighted residual statement for the continuity equation:

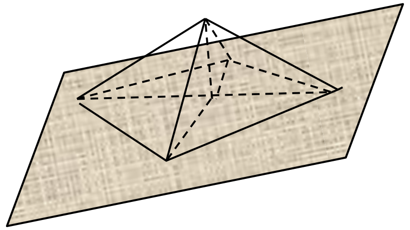
$$\int_{\Omega} \phi_i \frac{\eta^{n+1} - \eta^n}{\Delta t} d\Omega + \theta \left[-\int_{\Omega} \nabla \phi_i \cdot \boldsymbol{U}^{n+1} d\Omega + \int_{\Gamma} \phi_i U_n^{n+1} d\Gamma \right] + (1-\theta) \left[-\int_{\Omega} \nabla \phi_i \cdot \boldsymbol{U}^{n} d\Omega + \int_{\Gamma} \phi_i U_n^{n} d\Gamma \right] = 0,$$

$$(i = 1, \cdots, N_p)$$

ϕ_i: shape/weighting function; **U**: depth-integrated velocity; θ:implicitness factor

Need to eliminate $\mathbf{U}^{n+1}$ to get an equation for η alone. We'll do this with the aid from momentum equation:

$$\frac{\boldsymbol{u}^{\boldsymbol{n}+\mathbf{1}} - \boldsymbol{u}^*}{\Delta t} = \boldsymbol{f} - g\theta \nabla \eta^{n+1} - g(1-\theta)\nabla \eta^n + \boldsymbol{m}_{\boldsymbol{z}}^{\boldsymbol{n}+\mathbf{1}} - \alpha |\mathbf{u}| \boldsymbol{u}^{\boldsymbol{n}+\mathbf{1}} L(x,y,z)$$



Shape function (global)

2D case

The unknown velocity can be easily found as:

$$\boldsymbol{U}^{n+1} = \breve{\boldsymbol{G}} - g\theta\Delta t \frac{H^2}{\widetilde{H}} \nabla\eta^{n+1}$$

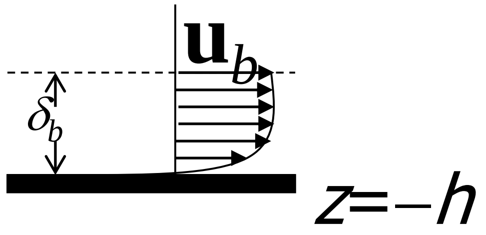
$$\breve{\boldsymbol{G}} = \frac{H}{\widetilde{H}}[\boldsymbol{U}^* + \Delta t(\boldsymbol{F} + \boldsymbol{\tau}_w - g(1-\theta)H\nabla\eta^n)]$$ (explicit terms)

$$\widetilde{H} = H + (\chi + \alpha|\boldsymbol{u}|H)\Delta t$$

3D case

Vertical integration of the momentum equation

$$\frac{\boldsymbol{U}^{n+1}-\boldsymbol{U}^{*}}{\Delta t}=\boldsymbol{F}-g\theta\nabla\eta^{n+1}-g(1-\theta)\nabla\eta^{n}+\boldsymbol{\tau}_w-\chi\boldsymbol{u}_b^{n+1}-\alpha\overline{|\boldsymbol{u}|}\boldsymbol{U}^{\boldsymbol{\alpha}}$$



Assuming: $$\int_{-h}^{z_v}|\boldsymbol{u}|\boldsymbol{u}dz \cong \overline{|\boldsymbol{u}|}\int_{-\boldsymbol{h}}^{\boldsymbol{z_v}}\boldsymbol{u}dz \equiv \overline{|\boldsymbol{u}|}\boldsymbol{U}^{\boldsymbol{\alpha}}$$

Momentum equation applied to the bottom boundary layer

$$\frac{\boldsymbol{u}_b^{n+1}-\boldsymbol{u}_b^{*}}{\Delta t}=\boldsymbol{f}_b-g\theta\nabla\eta^{n+1}-g(1-\theta)\nabla\eta^{n}-\alpha|\boldsymbol{u}_b|\boldsymbol{u}_b^{n+1}+\frac{\partial}{\partial z}\left(\nu\frac{\partial\boldsymbol{u}}{\partial z}\right)$$

Therefore

$$\boldsymbol{u}_b^{n+1}=\frac{1}{1+\alpha|\boldsymbol{u}_b|\Delta t}\left[\boldsymbol{u}_b^{*}+\boldsymbol{f}_b\Delta t-g(1-\theta)\Delta t\nabla\eta^{n}\right]-\frac{g\theta\Delta t}{1+\alpha|\boldsymbol{u}_b|\Delta t}\nabla\eta^{n+1}$$

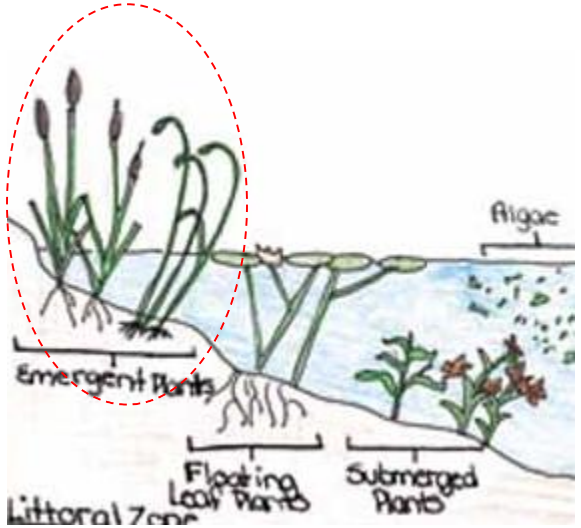
Now still need to find $\boldsymbol{U}^{\alpha}$

3D case: locally emergent vegetation

We have $\boldsymbol{U}^{\alpha} = \boldsymbol{U}^{\mathrm{n}+1}$

So:

$$\boldsymbol{U}^{n+1} = \boldsymbol{G_1} - \frac{g\theta\widehat{H}\Delta t}{1+\alpha\overline{|\boldsymbol{u}|}\Delta t}\nabla\eta^{n+1}$$



$$\boldsymbol{G}_1 = \frac{\boldsymbol{U}^* + (\boldsymbol{F}+\boldsymbol{\tau_w})\Delta t - g(1-\theta)\widehat{H}\Delta t\nabla\eta^n - \tilde{\chi}\Delta t(\boldsymbol{u}_b^* + \boldsymbol{f}_b\Delta t)}{1+\alpha\overline{|\boldsymbol{u}|}\Delta t}$$

$$\widehat{H} = H - \tilde{\chi}\Delta t$$

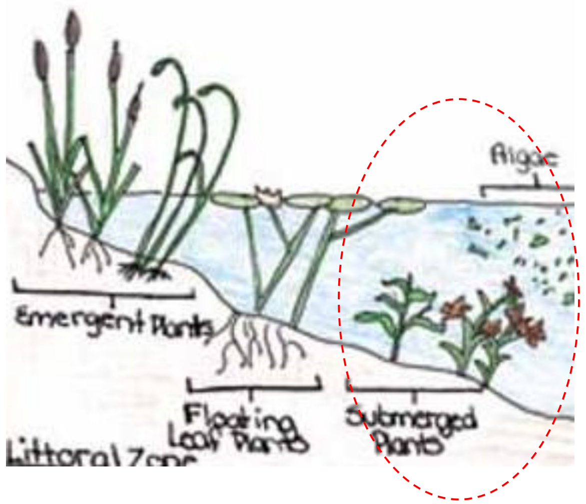
$$\tilde{\chi} = \frac{\chi}{1+\alpha|\boldsymbol{u_b}|\Delta t}$$

3D case: locally submerged vegetation

Integrating from bottom to top of canopy:

$$\boldsymbol{U}^{\alpha} = \boldsymbol{U}^{*\alpha} + \boldsymbol{F}^{\alpha}\Delta t - g\theta H^{\alpha}\Delta t\nabla\eta^{n+1} - g(1-\theta)H^{\alpha}\Delta t\nabla\eta^{n} - \alpha\Delta t\overline{|\boldsymbol{u}|}\boldsymbol{U}^{\alpha} + \Delta t\nu\left.\frac{\partial\boldsymbol{u}}{\partial z}\right|_{-h}^{z_v}$$

$$\boldsymbol{U}^{*\alpha} = \int_{-h}^{z_v}\boldsymbol{u}^{*}dz,\quad \boldsymbol{F}^{\alpha} = \int_{-h}^{z_v}\boldsymbol{f}dz$$



Inside vegetation layer, the stress obeys exponential law (Shimizu and Tsujimoto 1994)

$$\nu\frac{\partial\boldsymbol{u}}{\partial z} \equiv \overline{u'w'} = \boldsymbol{R}_0 e^{\beta_2(z-z_v)}, \qquad (z \le z_v)$$

β_2 is a constant found from empirical formula:

$$\beta_2 = \sqrt{\frac{\sqrt{N_v}}{H^{\alpha}}}\left[-0.32 - 0.85\log_{10}\left(\frac{H-H^{\alpha}}{H^{\alpha}}I\right)\right] \qquad I = \frac{\chi|\boldsymbol{u}_b|}{gH}$$

Therefore $\nu\left.\frac{\partial\boldsymbol{u}}{\partial z}\right|_{-h}^{z_v} = \beta\chi\boldsymbol{u}_b^{n+1}$ $\quad (\beta = e^{\beta_2(z_v-z_b)} - 1)$

$$\boldsymbol{U}^{\alpha} = \boldsymbol{G}_3 - \frac{g\theta\widehat{H}^{\alpha}\Delta t}{1+\alpha\overline{|\boldsymbol{u}|}\Delta t}\nabla\eta^{n+1} \qquad \boldsymbol{G}_3 = \frac{\boldsymbol{U}^{*\alpha} + \boldsymbol{F}^{\alpha}\Delta t + \beta\tilde{\chi}\Delta t(\boldsymbol{u}_b^{*} + \boldsymbol{f}_b\Delta t) - g(1-\theta)\widehat{H}^{\alpha}\Delta t\nabla\eta^{n}}{1+\alpha\overline{|\boldsymbol{u}|}\Delta t} \qquad \widehat{H}^{\alpha} = H^{\alpha} + \beta\tilde{\chi}\Delta t$$

Finally:

$$\boxed{\boldsymbol{U}^{n+1} = \boldsymbol{G}_2 - g\theta\overline{\overline{H}}\Delta t\nabla\eta^{n+1}}$$

$$\overline{\overline{H}} = H - \tilde{\chi}\Delta t - c\widehat{H}^{\alpha} \qquad c = \frac{\alpha\overline{|\boldsymbol{u}|}\Delta t}{1+\alpha\overline{|\boldsymbol{u}|}\Delta t}$$

General case

In compact form:

$$\boldsymbol{U}^{n+1} = \boldsymbol{E} - g\theta\breve{H}\Delta t\nabla\eta^{n+1}$$

$$\breve{H} = \begin{cases} \dfrac{H^2}{\widetilde{H}}, & \text{locally 2D} \\ \dfrac{\widehat{H}}{1+\alpha\overline{|\boldsymbol{u}|}\Delta t}, & \text{locally 3D emergent} \\ \overline{\overline{H}}, & \text{locally 3D submerged} \end{cases}$$

(Note the difference in $\breve{H}$ between 2D/3D)

$$\boldsymbol{E} = \begin{cases} \breve{\boldsymbol{G}}, & \text{2D} \\ \boldsymbol{G_1}, & \text{3D emergent} \\ \boldsymbol{G_2}, & \text{3D submerged} \end{cases}$$

Finite-element formulation (cont'd)

Finally, substituting this eq. back to continuity eq. we get one equation for elevations alone:

$$\underbrace{\int_{\Omega} \left[\phi_i \eta^{n+1} + g\theta^2 \Delta t^2 \breve{H} \nabla\phi_i \cdot \nabla\eta^{n+1}\right] d\Omega}_{I_1}$$

$$= \underbrace{\int_{\Omega} [\phi_i \eta^n + \theta\Delta t \nabla\phi_i \cdot \boldsymbol{E} + (1-\theta)\Delta t \nabla\phi_i \cdot \mathbf{U}^n] d\Omega}_{I_4} - \theta\Delta t \underbrace{\int_{\Gamma_v} \phi_i \widehat{U}_n^{n+1} d\Gamma_v}_{I_3} - (1-\theta)\,\Delta t \underbrace{\int_{\Gamma} \phi_i U_n^n d\Gamma}_{I_5} - \theta\Delta t \underbrace{\int_{\overline{\Gamma}_v} \phi_i U_n^{n+1} d\overline{\Gamma}_v}_{I_6} + \underbrace{I_7}_{\text{sources}}$$

(i=1,…,N_p)

Notes:

- When node i is on boundaries where essential b.c. is imposed, η is known and so no equations are needed, and so I_2 and I_6 need not be evaluated there
- When node i is on boundaries where natural b.c. is imposed, the velocity is known and so the last term on LHS is known → only the first term is truly unknown!
- b.c. is free lunch in FE model (cf. staggering in FV)
- The inherent numerical dispersion in FE nicely balances out the numerical diffusion inherent in the implicit time stepping scheme!

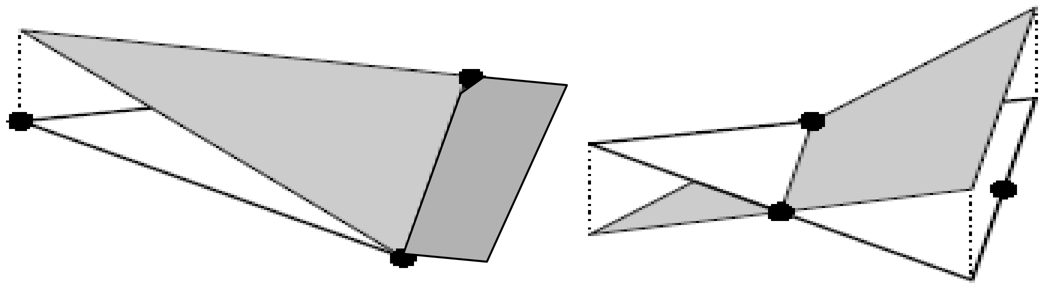
Shape function

- Used to approximate the unknown function
- Although usually used within each individual elements, shape function is global not local!

$$u = \sum_{i=1}^{N} u_i \phi_i, \quad \phi_i(x_j) = \delta_{ij}$$

- Must be sufficiently smooth to allow integration by part in weak formulation
- Mapping between local and global coordinates
- Assembly of global matrix

$$\int L\left(\sum_{i=1}^{N} u_i \phi_i\right)\phi_j d\Omega = \sum_{m=1}^{M} \int_{\Omega_m} L\left(\sum_{i=1}^{N} u_i \phi_i\right)\phi_j d\Omega_m = 0, \quad j = 1, \ldots, N$$

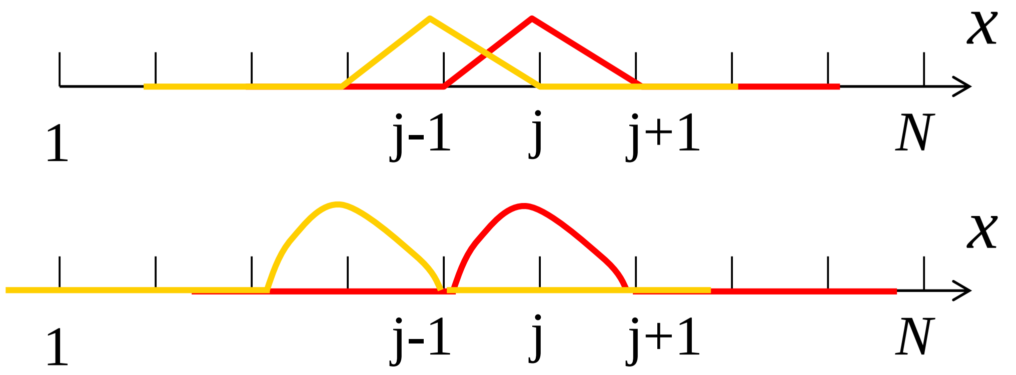


Conformal

Non-conformal

(Elevation, velocity)

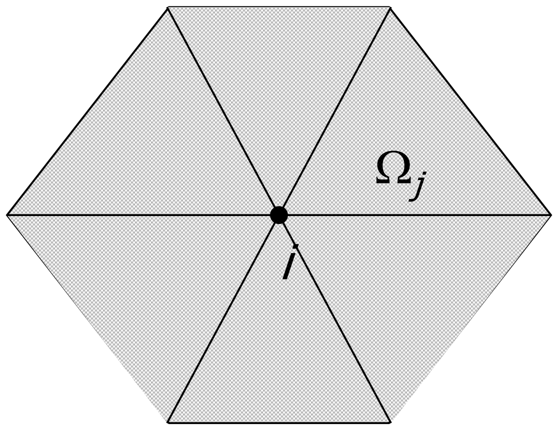
(optional for velocity: indvel=0 -> less dissipation and needs further stabilization (MB* schemes)



Matrices: I_1

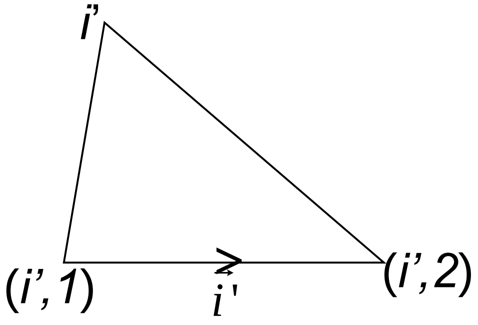
$$I_1 = \sum_{j=1}^{N_i} \int_{\Omega_j} (\hat{\phi}_{i'} \eta^{n+1} + g\theta^2 \Delta t^2 \breve{H} \nabla \hat{\phi}_{i'} \cdot \nabla \eta^{n+1}) d\Omega_j$$

$$= \sum_{j=1}^{N_i} \sum_{l=1}^{i34(j)} \eta_{j,l}^{n+1} \left[\frac{1+\delta_{i',l}}{12} A_j + \frac{g\theta^2 \Delta t}{4A_j} \overline{\breve{H}} \, \vec{i'} \cdot \vec{l} \right]$$



i' is local index of node *i* in Ω_j.

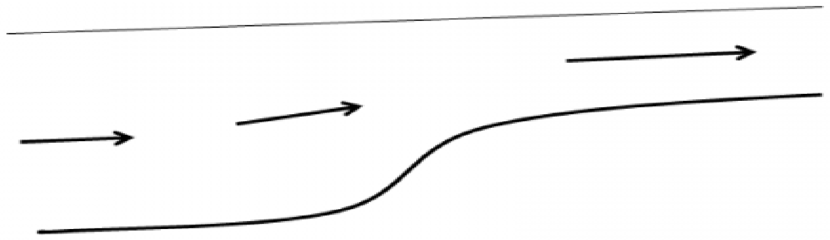
$\vec{i'} \cdot \vec{l}$ reaches its max when *i'*=*l*, and so does I_1.



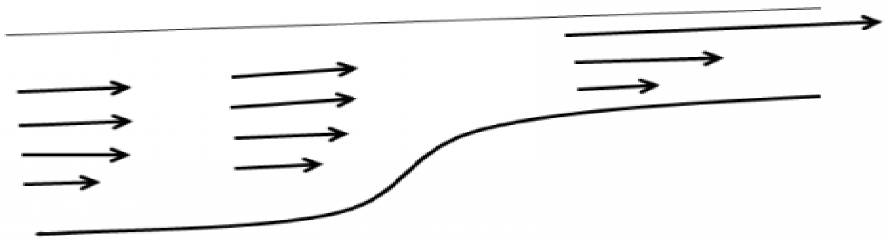
$$\sum_{i'=1}^{3} \vec{i'} = 0$$

so diagonal is dominant **if the averaged friction-reduced depth ≥0** (can be relaxed)

Mass matrix is positive definite and symmetric! JCG or PETSc solvers work fine



(a) 2D model velocity

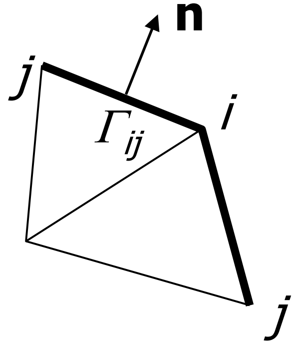


(b) 3D model velocity

Matrices: I_3 and I_5

$$I_3 = \int_{\Gamma_v} \phi_i \hat{U}_n^{n+1} d\Gamma_v = \sum_j \int_{\Gamma_{ij}} \phi_i \hat{U}_n^{n+1} d\Gamma_{ij}$$

$$= \sum_j \frac{L_{ij}}{4} \sum_{k=k_{bs}}^{N_z-1} \Delta z_{ij,k+1} \left[(\hat{u}_{\mathrm{n}}^{n+1})_{ij,k+1} + (\hat{u}_{\mathrm{n}}^{n+1})_{ij,k} \right]$$



Flather b.c. (overbar denotes mean)

$$\hat{u}_{\mathrm{n}}^{n+1} - \overline{u}_{\mathrm{n}} = \sqrt{g/h}(\eta^{n+1} - \overline{\eta})$$

$$I_3 = \sum_j L_{ij} \left\{ \frac{(\overline{U}_{\mathrm{n}})_{ij}}{2} + \frac{\sqrt{gh_{ij}}}{6} \left[2(\eta_i^{n+1} - \overline{\eta}_i) + (\eta_j^{n+1} - \overline{\eta}_j) \right] \right\}$$

Similarly

$$I_5 = \sum_j \int_{\Gamma_{ij}} \hat{\phi}_i \hat{U}_n^n d\Gamma_{ij} = \sum_j \frac{L_{ij}}{4} \sum_{k=k_{bs}}^{N_z-1} \Delta z_{ij,k+1} \left[(\hat{u}_{\mathrm{n}}^n)_{ij,k+1} + (\hat{u}_{\mathrm{n}}^n)_{ij,k} \right]$$

Matrices: I_7

Related to source/sink

$$I_7 = \frac{\Delta t}{3} \sum_{j \in B_i \cap (\cup \Omega_m)} s_j(t), \;\; (i = 1, \ldots, N_p)$$

B_i: ball of node *i*
$U\Omega_m$: elements that have source/sink
S_j: volume discharge rate [m^3/s]

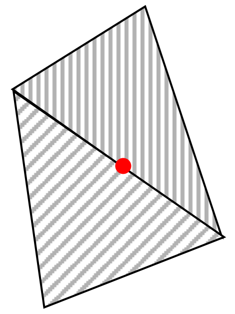
Matrices: I_4

Has the most complex form

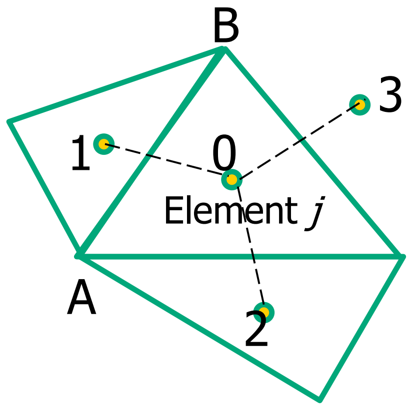
$$I_3 = \int_\Omega \left[\phi_i \eta^n + (1-\theta)\Delta t \nabla \phi_i \cdot \mathbf{U}^n + \theta \Delta t \nabla \phi_i \cdot (\boldsymbol{E} \right] d\Omega = \sum_j^{N_i} \left[\frac{A_j}{12} \sum_{l=1}^{i34(j)} \eta^n_{j,l} (1 + \delta_{i',l}) + (1-\theta)\Delta t A_j \nabla \hat{\phi}_{i'} \cdot \bar{\mathbf{U}}^n_j + \theta \Delta t \nabla \hat{\phi}_i \cdot \hat{\mathbf{G}}'_j \right]$$

Most complex part of ***E*** is the baroclinicity

$$\mathbf{f}_c = -\frac{g}{\rho_0} \int_z^\eta \nabla \rho \, d\zeta$$ at elements/sides



Baroclinicity: *z*-method



- Interpolate density along horizontal planes
  - Advantage: alleviate pressure gradient errors (Fortunato and Baptista 1996)
  - Disadvantage: near surface or bottom
- Density is defined at prism centers (half levels)
- Re-construction method: compute directional derivatives along two direction at node *i* and then convert them back to *x,y*
- Density gradient calculated at prism centers first (for continuity eq.); the values at face centers are averaged from those at prism centers (for momentum eq.)
  - constant extrapolation (shallow) is used near bottom (under resolution)
  - Cubic spline is used in all interpolations of ρ
  - Mean density profile $\overline{\rho}(z)$ can be optionally removed
- 3 or 4 eqs for the density gradient vs. 2 unknowns – averaging needed for 3 pairs; e.g.

$$\begin{cases} (\rho_x^{'})_{i,k}(x_1 - x_0) + (\rho_y^{'})_{i,k}(y_1 - y_0) = \rho_1^{'} - \rho_0^{'} \\ (\rho_x^{'})_{i,k}(x_2 - x_0) + (\rho_y^{'})_{i,k}(y_2 - y_0) = \rho_2^{'} - \rho_0^{'} \end{cases}$$

  - Degenerate cases: when the two vectors are co-linear (discard)
  - Vertical integration using trapzoidal rule

Momentum equation: 2D case

The depth-averaged velocity can be directly solved once unknown elevations are found:

$$\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}} = \frac{\breve{H}}{H}\left[\boldsymbol{u}^{*} + (\boldsymbol{f} + \boldsymbol{\tau}_{\boldsymbol{w}}/H)\Delta t - g\theta\nabla\eta^{n+1} - g(1-\theta)\nabla\eta^{n}\right]$$

Momentum equation: 3D case

Galerkin FE in the vertical

$$\int_{z_b}^{\eta} \varphi_l \left[b\boldsymbol{u}^{n+1} - \Delta t \frac{\partial}{\partial z}\left(\nu \frac{\partial \boldsymbol{u}^{n+1}}{\partial z} \right) \right] dz = \int_{z_b}^{\eta} \varphi_l \boldsymbol{g} dz, \qquad (l = k_b + 1, \cdots, N_z)$$

$$b = 1 + \alpha |\boldsymbol{u}^n| \Delta t \mathcal{H}(z_v - z)$$

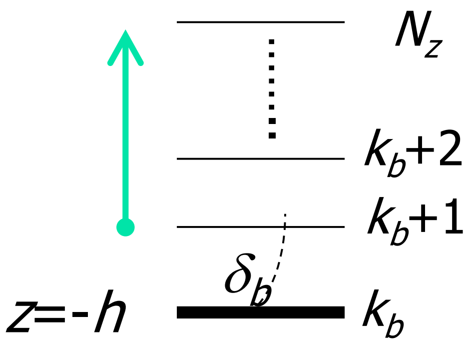
φ_l: vertical hat function; $\boldsymbol{g}$: explicit terms

$$\mathrm{A}(N_z - l)\left[\frac{b\Delta z_{l+1}}{6}(2\mathbf{u}_l^{n+1} + \mathbf{u}_{l+1}^{n+1}) - \nu_{l+1/2}\Delta t \frac{\mathbf{u}_{l+1}^{n+1} - \mathbf{u}_l^{n+1}}{\Delta z_{l+1}} \right] + \mathrm{A}(k_b + 1 - l)\left[\frac{b\Delta z_l}{6}(2\mathbf{u}_l^{n+1} + \mathbf{u}_{l-1}^{n+1}) + \nu_{l-1/2}\Delta t \frac{u_l^{n+1} - u_{l-1}^{n+1}}{\Delta z_l} \right] + \delta_{l,k_b+1}\Delta t \chi \mathbf{u}_{k_b+1}^{n+1} =$$

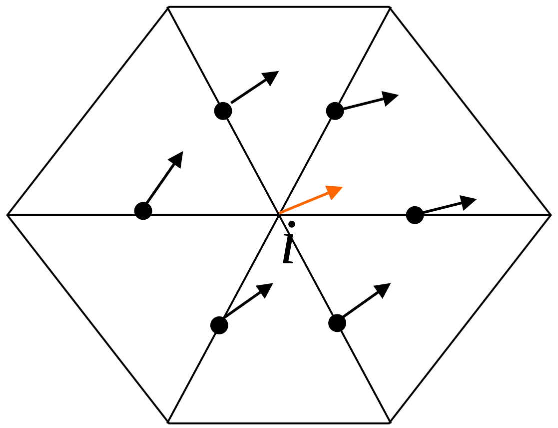
$$\delta_{l,N_z}\Delta t \boldsymbol{\tau}_w^{n+1} + \mathrm{A}(N_z - l)\frac{\Delta z_{l+1}}{6}(2\mathbf{g}_l + \mathbf{g}_{l+1}) + \mathrm{A}(k_b + 1 - l)\frac{\Delta z_l}{6}(2\mathbf{g}_l^{n+1} + \mathbf{g}_{l-1}^{n+1})$$

$$(l = k_b + 1, \ldots N_z)$$

- The matrix is tri-diagonal (Neumann b.c. can be imposed easily)
- Solution at bottom from the b.c. there (**u**=0 for $\chi \neq 0$); otherwise free slip
- Vegetation terms are also treated implicitly

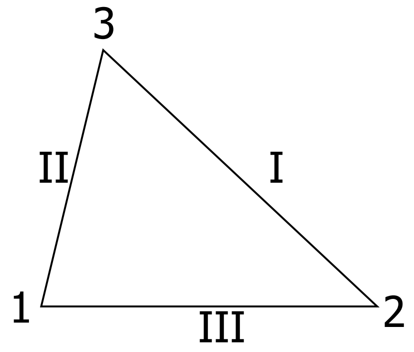


Velocity at nodes



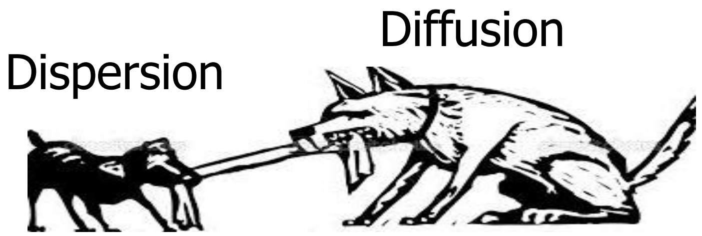
• Needed for ELM (interpolation), and can serve as a way to further stabilize the mom solver by adding some inherent numerical diffusion
• Method 1: inverse-distance averaging around ball (*indvel*=1 or MA schemes) – larger dissipation; no further stabilization needed
• Method 2: use linear shape function (conformal) (*indvel*=0 or MB schemes)

$$\mathbf{u}_1 = \mathbf{u}_{II} + \mathbf{u}_{III} - \mathbf{u}_I$$



• Discontinuous across elements; averaging to get the final value at node
• Generally needs velocity b.c. (for incoming info)
• indvel=0 generally gives less dissipation but needs further stabilization in the form of viscosity or Shapiro filter for stabilization (important for eddying regime)

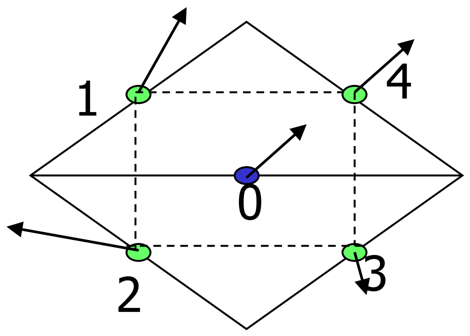
Eternal struggle between diffusion and dispersion



Shaprio filter

A simple filter to reduce sub-grid scale (unphysical) oscillations while leaving the physical signals largely intact (α=0.5 optimal)

$$\hat{\mathbf{u}}_0 = \mathbf{u}_0 + \frac{\alpha}{4}\left(\sum_{i=1}^{4}\mathbf{u}_i - 4\mathbf{u}_0\right), \quad (0 \le \alpha \le 0.5)$$



- No filtering for boundary sides – so need b.c. there especially for incoming flow
- Equivalent to Laplacian viscosity

Horizontal viscosity

Important for *eddying regime* for adding proper dissipation

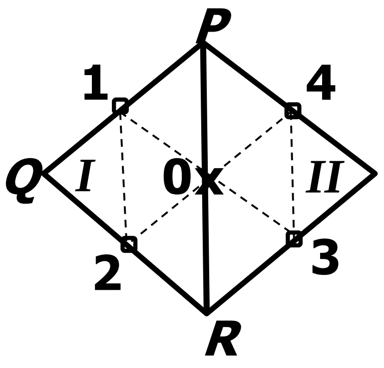
$$\nabla\cdot(\mu\nabla u)\big|_0 \cong \frac{\mu_0}{A_I + A_{II}}\oint_\Gamma \frac{\partial u}{\partial n}\, d\Gamma$$

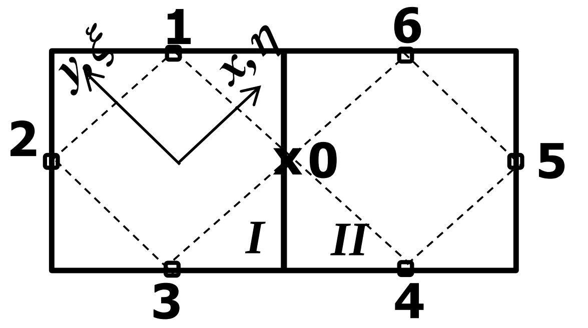
Assuming uniformity

⬇

(like Shapiro filter)

$$\nabla\cdot(\mu\nabla u)\big|_0 = \frac{\mu_0}{\sqrt{3}A_I}(u_1+u_2+u_3+u_4-4u_0)$$

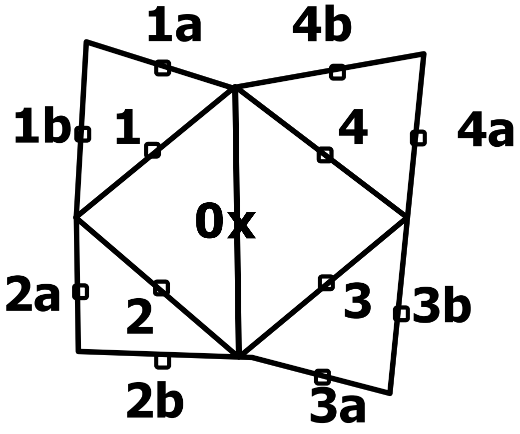




Bi-harmonic viscosity (ideal in eddying regime)

$$-\lambda\nabla^4 u\big|_0 = -\lambda\gamma_3(\nabla^2 u_1+\nabla^2 u_2+\nabla^2 u_3+\nabla^2 u_3-4\nabla^2 u_0) =$$

$$\frac{\gamma_2}{\Delta t}\left[7(u_1+u_2+u_3+u_4)-u_{1a}-u_{1b}-u_{2a}-u_{2b}-u_{3a}-u_{3b}-u_{4a}-u_{4b}-20u_0\right]$$



(Zhang et al. 2016)

Vertical velocity

- Vertical velocity using Finite Volume

$$\hat{S}_{k+1}(\bar{u}_{k+1}^{n+1}n_{k+1}^{x}+\bar{v}_{k+1}^{n+1}n_{k+1}^{y}+w_{i,k+1}^{n+1}n_{k+1}^{z})-\hat{S}_{k}(\bar{u}_{k}^{n+1}n_{k}^{x}+\bar{v}_{k}^{n+1}n_{k}^{y}+w_{i,k}^{n+1}n_{k}^{z})+$$
$$\sum_{m=1}^{3}\hat{P}_{js(i,m)}s_{i,m}(\hat{q}_{js(i,m),k}^{n+1}+\hat{q}_{js(i,m),k+1}^{n+1})/2=0,\quad (k=k_b,\dots,N_z-1)$$

$$\hat{q}_{j,k}=\mathbf{u}_{j,k}\cdot\mathbf{n}_j$$

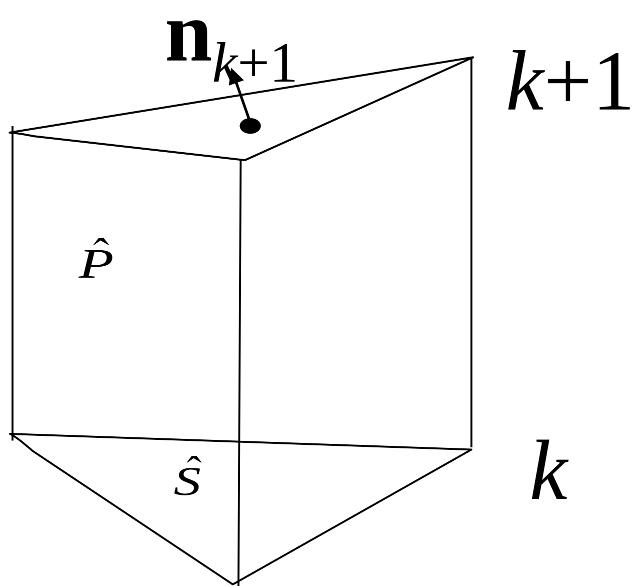
i34(j)

$$w_{i,k+1}^{n+1}=\frac{1}{n_{k+1}^{z}\hat{S}_{k+1}}\left[-\sum_{m=1}^{3}\hat{P}_{js(i,m)}s_{i,m}(\hat{q}_{js(i,m),k}^{n+1}+\hat{q}_{js(i,m),k+1}^{n+1})/2-\hat{S}_{k+1}(\bar{u}_{k+1}^{n+1}n_{k+1}^{x}+\bar{v}_{k+1}^{n+1}n_{k+1}^{y})-\hat{S}_{k}(\bar{u}_{k}^{n+1}n_{k}^{x}+\bar{v}_{k}^{n+1}n_{k}^{y}+w_{i,k}^{n+1}n_{k}^{z})\right],$$
$$(k=k_b,\dots,N_z-1)$$

In compact form:

$$\sum_{j\in S^{+}}|Q_j|=\sum_{j\in S^{-}}|Q_j|$$

This is the foundation of mass conservation and monotonicity in transport solvers



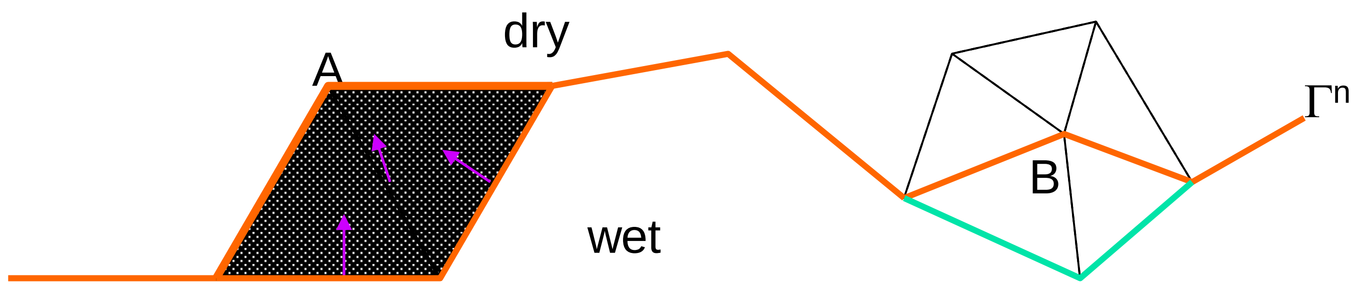
Bottom b.c.

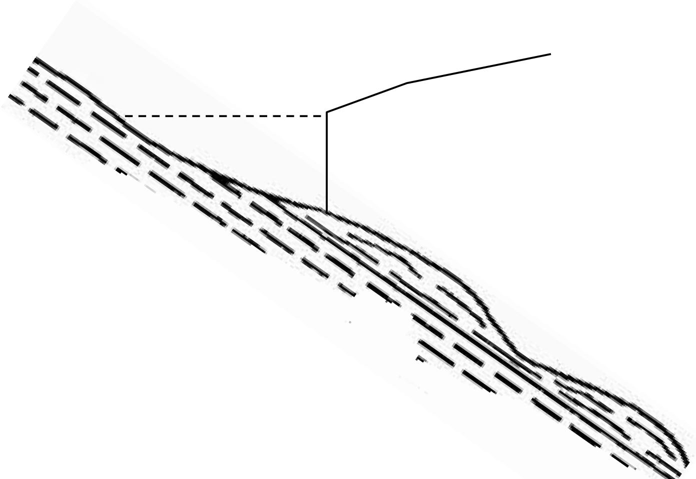
$$\bar{u}_{k}^{n+1}n_{k}^{x}+\bar{v}_{k}^{n+1}n_{k}^{y}+w_{i,k}^{n+1}n_{k}^{z}=\frac{\partial b}{\partial t},\quad (k=k_b)$$

Surface b.c. not enforced (closure error)

Inundation algorithms

- Algorithm 1 (*inunfl*=0)
  - Update wet and dry elements, sides, and nodes at the end of each time step based on the newly computed elevations. Newly wetted vel., S,T etc calculated using average
- Algorithm 2 (*inunfl*=1; accurate inundation for wetting and drying with sufficient grid resolution)
  - Wet → add elements & extrapolate velocity
  - Dry → remove elements
  - Iterate until the new interface is found at step n+1
  - Extrapolate elevation at final interface (smoothing effects)



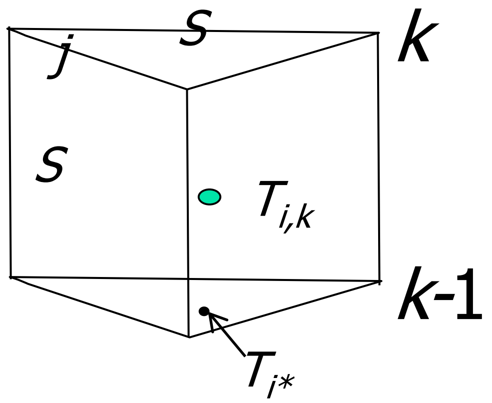


Extrapolation of surface

Transport: upwind (1)

$$\begin{cases} \dfrac{\partial T}{\partial t} + \nabla_3 \cdot (\mathbf{u}T) = \dfrac{\partial}{\partial z}\left(\kappa \dfrac{\partial T}{\partial z}\right) + \hat{Q} + \nabla \cdot (\kappa_h \nabla T) \\ \kappa \dfrac{\partial T}{\partial z} = \hat{T}, \quad z = \eta \\ \kappa \dfrac{\partial T}{\partial n} = 0, \quad z = -h \end{cases}$$

(sources: bdy_frc, flx_sf and flx_bt)



Finite volume discretization in a prism (i,k): mass conservation

i34(j)+2

$$V_i \frac{T_i^{m+1} - T_i^m}{\Delta t'} + \sum_{j \in \partial V}^{5} u_j S_j T_{j*} = V_i \hat{Q}_{i,k}^m + A_i \Delta t' \left[\kappa_{i,k} \frac{T_{i,k+1}^{m+1} - T_{i,k}^{m+1}}{\Delta z_{i,k+1/2}} - \kappa_{i,k-1} \frac{T_{i,k}^{m+1} - T_{i,k-1}^{m+1}}{\Delta z_{i,k-1/2}} \right] +$$

i34(j)

$$\Delta t' \sum_{j=1}^{3} (\kappa_h)_j \frac{T_j^m - T_i^m}{\delta_{ij}} S_j, \quad (k = k_b + 1, \cdots, N_z)$$

$m \neq n$, $\Delta t' \neq \Delta t$;
$T_i = T_{i,k}$;
u_j is outward normal velocity

Focus on the advection first

$$T_i^{m+1} = T_i^m - \frac{\Delta t'}{V_i^n} \sum_{j \in \partial V} Q_j T_{j*} \qquad Q_j \equiv u_j S_j$$

From continuity equation

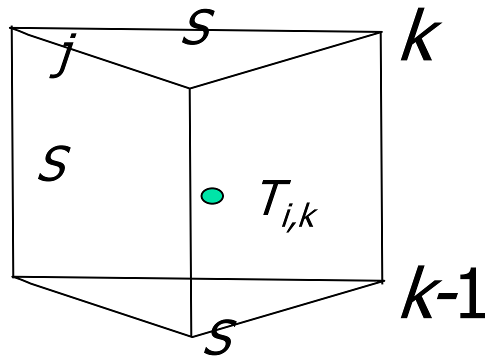
$$\sum_j Q_j = 0 \Rightarrow \sum_{j \in S^+} |Q_j| = \sum_{j \in S^-} |Q_j|$$

S^+: outflow faces; S^-: inflow faces

Upwinding

$$T_{j*} = \begin{cases} T_i, & j \in S^+ \\ T_j, & j \in S^- \end{cases}$$

Transport: upwind (2)



Drop "m" for brevity (and keep only advection terms)

$$T_i^{m+1} = T_i - \frac{\Delta t'}{V_i^n}\left[\sum_{j\in S^+} |Q_j| T_i - \sum_{j\in S^-} |Q_j| T_j\right] = \left(1 - \frac{\Delta t'}{V_i}\sum_{j\in S^+} |Q_j|\right) T_i + \frac{\Delta t'}{V_i}\sum_{j\in S^-} |Q_j| T_j =$$
$$(1-\alpha)T_i + \alpha T_j$$

Max. principle is guaranteed if

$$1 - \frac{\Delta t'}{V_i}\sum_{j\in S^-} |Q_j| \geq 0 \Rightarrow \Delta t' \leq \frac{V_i}{\sum_{j\in S^-} |Q_j|}$$

Courant number condition (3D case)
Global time step for all prisms

Implicit treatment of vertical advective fluxes to bypass vertical Courant number restriction in shallow depths

$$\Delta t' \leq \frac{V_i}{\sum_{j\in S^-_H} |Q_j|}$$

(Modified Courant number condition)

Mass conservation

- Budget – vertical and horizontal sums
- movement of free surface: small conservation error
- Includes source/sink

Solar radiation

total solar radiation

$$\hat{Q} = \frac{1}{\rho_0 C_p}\frac{\partial \dot{H}}{\partial z}, \quad \dot{H}(z) = \dot{H}(0)\left[\mathrm{R}e^{-D/d_1} + (1-\mathrm{R})e^{-D/d_2}\right]$$

Budget

$$\int_{-\infty}^{0} \hat{Q}dz = \frac{\dot{H}(0)}{\rho_0 C_p}$$

The source term in the upwind scheme (F.D.)

$$V_i \hat{Q}_i^m = \frac{V_i}{\rho_0 C_p}\frac{\dot{H}_{i,k}^m - \dot{H}_{i,k-1}^m}{\Delta z_{i,k}} = \frac{A_i(\dot{H}_{i,k}^m - \dot{H}_{i,k-1}^m)}{\rho_0 C_p}$$

*Assume the heat is completely absorbed by the bed (black body) to prevent overheating in shallow depths

Transport: TVD[2]

Solve the transport eq in 3 sub-steps:

$$
\begin{cases}
C_i^{m+1} = C_i^n + \dfrac{\Delta t_m}{V_i} \sum_{j \in S_H^-} |Q_j| (C_j^m - C_i^m) - \dfrac{\Delta t_m}{V_i} \sum_{j \in S_H} Q_j \hat{\psi}_j^m, \quad (m = 1, \ldots, M) \\
\tilde{C}_i = C_i^{M+1} + \dfrac{\Delta t}{V_i} \sum_{j \in S_V^-} |Q_j| (\tilde{C}_j - \tilde{C}_i) - \dfrac{\Delta t}{V_i} \sum_{j \in S_V} Q_j (\Phi_j + \Psi_j), (j = k_b, \ldots, N_z) \\
C_i^{n+1} = \tilde{C}_i + \dfrac{A_i \Delta t}{V_i} \left[\left(\kappa \dfrac{\partial C}{\partial z} \right)_{i,k}^{n+1} - \left(\kappa \dfrac{\partial C}{\partial z} \right)_{i,k-1}^{n+1} \right] + \dfrac{\Delta t}{V_i} \int_{V_i} F_h^n dV, (k = k_b, \ldots, N_z)
\end{cases}
$$

Horizontal advection (explicit TVD method or WENO)

Vertical advection (implicit TVD[2])

Remaining (implicit diffusion)

- 1st step: explicit TVD or WENO[3]
- 2nd step: Taylor expansion in space *and time*

$$
C_j^{n+1/2} = C_{jup}^{n+1} + \Phi_j + \Psi_j = C_{jup}^{n+1} + \mathbf{r} \bullet [\nabla C]_{jup}^{n+1} - \frac{\Delta t}{2} \left[\frac{\partial C}{\partial t} \right]_{jup}^{n+1}
$$

Space limiter (Φ_j)

Time limiter (Ψ_j)

Duraisamy and Baeder (2007)

IMPORTANT: built-in monotonicity constraint is very important for higher-order transport; use of additional diffusion would degrade order of accuracy

TVD[2]

$$\tilde{C}_i + \frac{\frac{\Delta t}{V_i}\sum_{j\in S_V^-}|Q_j|\left[1+\frac{1}{2}\left(\sum_{p\in S_V^+}\frac{\phi_p}{r_p}-\phi_j\right)\right](\tilde{C}_i-\tilde{C}_j)}{1+\frac{\Delta t}{2V_i}\sum_{j\in S_V^+}|Q_j|\left(\sum_{q\in S_V^-}\frac{\psi_q}{s_q}-\psi_j\right)} = C_i^{M+1}$$

Upwind ratio

$$r_p = \frac{\sum_{q\in S_V^-}|Q_q|(\tilde{C}_q-\tilde{C}_i)}{|Q_p|(\tilde{C}_i-\tilde{C}_p)},\ p\in S_V^+$$

Downwind ratio

$$s_q = \frac{(\tilde{C}_i-C_i^{M+1})\sum_{p\in S_V^+}|Q_p|}{|Q_q|(\tilde{C}_q-C_q^{M+1})},\ q\in S_V^-$$

TVD condition:

$$1+\frac{1}{2}\left(\sum_{p\in S_V^+}\frac{\phi_p}{r_p}-\phi_j\right)\geq 0$$

$$1+\frac{\Delta t}{2V_i}\sum_{j\in S_V^+}|Q_j|\left(\sum_{q\in S_V^-}\frac{\psi_q}{s_q}-\psi_j\right)\geq\delta>0$$

(iterative solver)

- Iterative solver converges very fast
- 2nd-order accurate in space *and time*
- Monotone
- Alleviate the sub-cycling/small Δt for transport that plagued explicit TVD
- Can be used at any depths!

Turbulence closure (itur=3)

$$\begin{cases} \dfrac{Dk}{Dt} = \dfrac{\partial}{\partial z}\left(\nu_k^{\psi}\dfrac{\partial k}{\partial z}\right) + \nu M^2 + \kappa N^2 + c_{fk}\alpha|\mathbf{u}|^3\mathcal{H}(z_v - z) - \epsilon \\ \dfrac{D\psi}{Dt} = \dfrac{\partial}{\partial z}\left(\nu_{\psi}\dfrac{\partial \psi}{\partial z}\right) + \dfrac{\psi}{k}\left[c_{\psi 1}\nu M^2 + c_{\psi 3}\kappa N^2 + c_{f\psi}\alpha|\mathbf{u}|^3\mathcal{H}(z_v - z) - c_{\psi 2}\epsilon F_{wall}\right] \end{cases}$$

Essential b.c.

$$\begin{cases} k = \dfrac{1}{(c_\mu^0)^2}\nu\left|\dfrac{\partial \mathbf{u}}{\partial z}\right|, \quad z = -h \text{ or } \eta \\ l = \kappa_0 \Delta, \quad z = -h \text{ or } \eta \\ \psi = (c_\mu^0)^p k^m (\kappa_0 \Delta)^n, \quad z = -h \text{ or } \eta \end{cases}$$

Natural b.c.

$$\begin{cases} \nu_k^{\psi}\dfrac{\partial k}{\partial z} = 0, \quad z = -h \text{ or } \eta \\ \nu_{\psi}\dfrac{\partial \psi}{\partial z} = \kappa_0 n \nu_{\psi}\dfrac{\psi}{l}, \quad z = -h \\ \nu_{\psi}\dfrac{\partial \psi}{\partial z} = -\kappa_0 n \nu_{\psi}\dfrac{\psi}{l}, \quad z = \eta \end{cases}$$

FE formulation

- k, ψ and diffusivities are defined at nodes and whole levels
- Use natural b.c. first and essential b.c. is used to overwrite the solution at boundary
- Neglect advection
- The production and buoyancy terms are treated explicitly or implicitly depending on the sum to enhance stability

Turbulence closure (itur=3)

$$\mathrm{A}(l-N_z)\left[\frac{\Delta z_{l+1}}{6}\left(2k_l^{n+1}+k_{l+1}^{n+1}-2k_l^{n}-k_{l+1}^{n}\right)-(\nu_k^{\psi})_{l+1/2}\Delta t\,\frac{k_{l+1}^{n+1}-k_l^{n+1}}{\Delta z_{l+1}}\right]+$$

$$\mathrm{A}(l-k_b)\left[\frac{\Delta z_l}{6}\left(2k_l^{n+1}+k_{l-1}^{n+1}-2k_l^{n}-k_{l-1}^{n}\right)+(\nu_k^{\psi})_{l-1/2}\Delta t\,\frac{k_l^{n+1}-k_{l-1}^{n+1}}{\Delta z_l}\right]=$$

$$\mathrm{A}(l-N_z)\Delta t\left\{\left[\begin{array}{c}\frac{\Delta z_{l+1}}{2}\left(\quad P \quad \right)_{l+1/2}^{n} \\ \frac{\Delta z_{l+1}}{6k_{l+1/2}^{n}}\left(\quad P \quad \right)_{l+1/2}^{n}(2k_l^{n+1}+k_{l+1}^{n+1})\end{array}\right]-(c_\mu^0)^3\left(k^{1/2}l^{-1}\right)_{l+1/2}^{n}\frac{\Delta z_{l+1}}{6}(2k_l^{n+1}+k_{l+1}^{n+1})\right\}+$$

$$\mathrm{A}(l-k_b)\Delta t\left\{\left[\begin{array}{c}\frac{\Delta z_{l}}{2}\left(\quad P \quad \right)_{l-1/2}^{n} \\ \frac{\Delta z_{l}}{6k_{l-1/2}^{n}}\left(\quad P \quad \right)_{l-1/2}^{n}(2k_l^{n+1}+k_{l-1}^{n+1})\end{array}\right]-(c_\mu^0)^3\left(k^{1/2}l^{-1}\right)_{l-1/2}^{n}\frac{\Delta z_{l}}{6}(2k_l^{n+1}+k_{l-1}^{n+1})\right\}$$

$$(l=k_b,\ldots,N_z)$$

Turbulence closure

$$A(l-N_z)\left[\frac{\Delta z_{l+1}}{6}\left(2\psi_l^{n+1}+\psi_{l+1}^{n+1}-2\psi_l^{n}-\psi_{l+1}^{n}\right)-(\nu_\psi)_{l+1/2}\Delta t\,\frac{\psi_{l+1}^{n+1}-\psi_l^{n+1}}{\Delta z_{l+1}}\right]+$$

$$A(l-k_b)\left[\frac{\Delta z_{l}}{6}\left(2\psi_l^{n+1}+\psi_{l-1}^{n+1}-2\psi_l^{n}-\psi_{l-1}^{n}\right)+(\nu_\psi)_{l-1/2}\Delta t\,\frac{\psi_{l}^{n+1}-\psi_{l-1}^{n+1}}{\Delta z_{l}}\right]=$$

$$A(l-N_z)\Delta t\left\{\begin{array}{l}\left[\begin{array}{l}\frac{\Delta z_{l+1}}{2}\left(\quad Q \quad\right)_{l+1/2}^{n}\left(\frac{\psi}{k}\right)_{l+1/2}^{n}\\ \frac{\Delta z_{l+1}}{6k_{l+1/2}^{n}}\left(\quad Q \quad\right)_{l+1/2}^{n}\left(2\psi_l^{n+1}+\psi_{l+1}^{n+1}\right)\end{array}\right]-\\ (c_\mu^0)^3\left(c_{\psi 2}k^{1/2}l^{-1}F_{wall}\right)_{l+1/2}^{n}\frac{\Delta z_{l+1}}{6}\left(2\psi_l^{n+1}+\psi_{l+1}^{n+1}\right)\end{array}\right\}+$$

$$A(l-k_b)\Delta t\left\{\begin{array}{l}\left[\begin{array}{l}\frac{\Delta z_{l}}{2}\left(\quad Q \quad\right)_{l-1/2}^{n}\left(\frac{\psi}{k}\right)_{l-1/2}^{n}\\ \frac{\Delta z_{l}}{6k_{l-1/2}^{n}}\left(\quad Q \quad\right)_{l-1/2}^{n}\left(2\psi_l^{n+1}+\psi_{l-1}^{n+1}\right)\end{array}\right]-\\ (c_\mu^0)^3\left(c_{\psi 2}k^{1/2}l^{-1}F_{wall}\right)_{l-1/2}^{n}\frac{\Delta z_{l}}{6}\left(2\psi_l^{n+1}+\psi_{l-1}^{n+1}\right)\end{array}\right\}$$

$$(l=k_b,\ldots,N_z)$$

GOTM turbulence library (itur=4)

- General Ocean Turbulence Model
- One-dimensional water column model for marine and limnological applications
- Coupled to a choice of traditional as well as state-of-the-art parameterizations for vertical turbulent mixing (including KPP)
- Some perplexing problems on some platforms – robustness?
  - Bugs page
  - Division by 0? Newer versions may have resolved this issue
- Works better than *itur*=3 otherwise

www.gotm.net

Radiation stress

$$\frac{Du}{Dt} = \dots - \frac{1}{\rho H}\frac{\partial H S_{xx}}{\partial x} - \frac{1}{\rho H}\frac{\partial H S_{xy}}{\partial y}$$

$$\frac{Dv}{Dt} = \dots - \frac{1}{\rho H}\frac{\partial H S_{yy}}{\partial y} - \frac{1}{\rho H}\frac{\partial H S_{xy}}{\partial x}$$

S_{xx} etc are functions of wave energy, which is integrated over all freq. and direction

WBBL (Grant-Madsen)

Wave-current friction factor

Ratio of shear stresses

$$\begin{cases} \mu & = \frac{\tau_b}{\tau_w} \\ c_\mu & = (1 + 2\mu|\cos\varphi_{wc}| + \mu^2)^{1/2} \\ f_w & = c_\mu exp[5.61(\frac{c_\mu U_w}{k_N \omega})^{-0.109} - 7.3] \\ \tau_w & = 0.5 f_w U_w^2 \end{cases}$$

φ_{wc} angle between bottom current and dominant wave direction

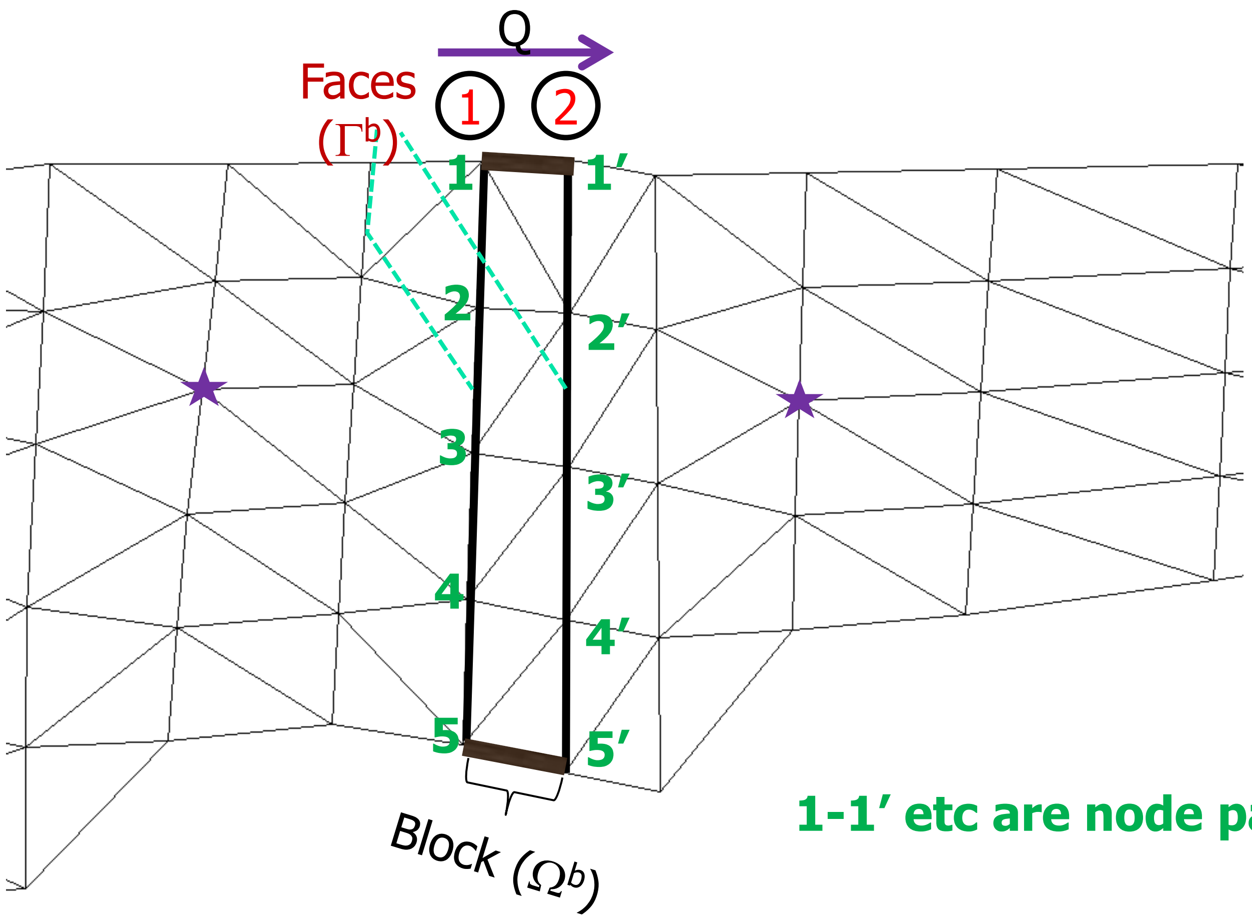
τ_b: current induced bottom stress

U_w: orbital vel. amplitude

ω: representative angular freq.

The nonlinear eq. system is solved with a simple iterative scheme starting from μ=0 (pure wave), c_μ=1. Strong convergence is observed for practical applications

Hydraulics



1-1′ etc are node pairs

The 2 'upstream' and 'downstream' reference nodes on each side of the structure are used to calculate the thru flow Q

The differential-integral eq. for continuity eq. is modified as

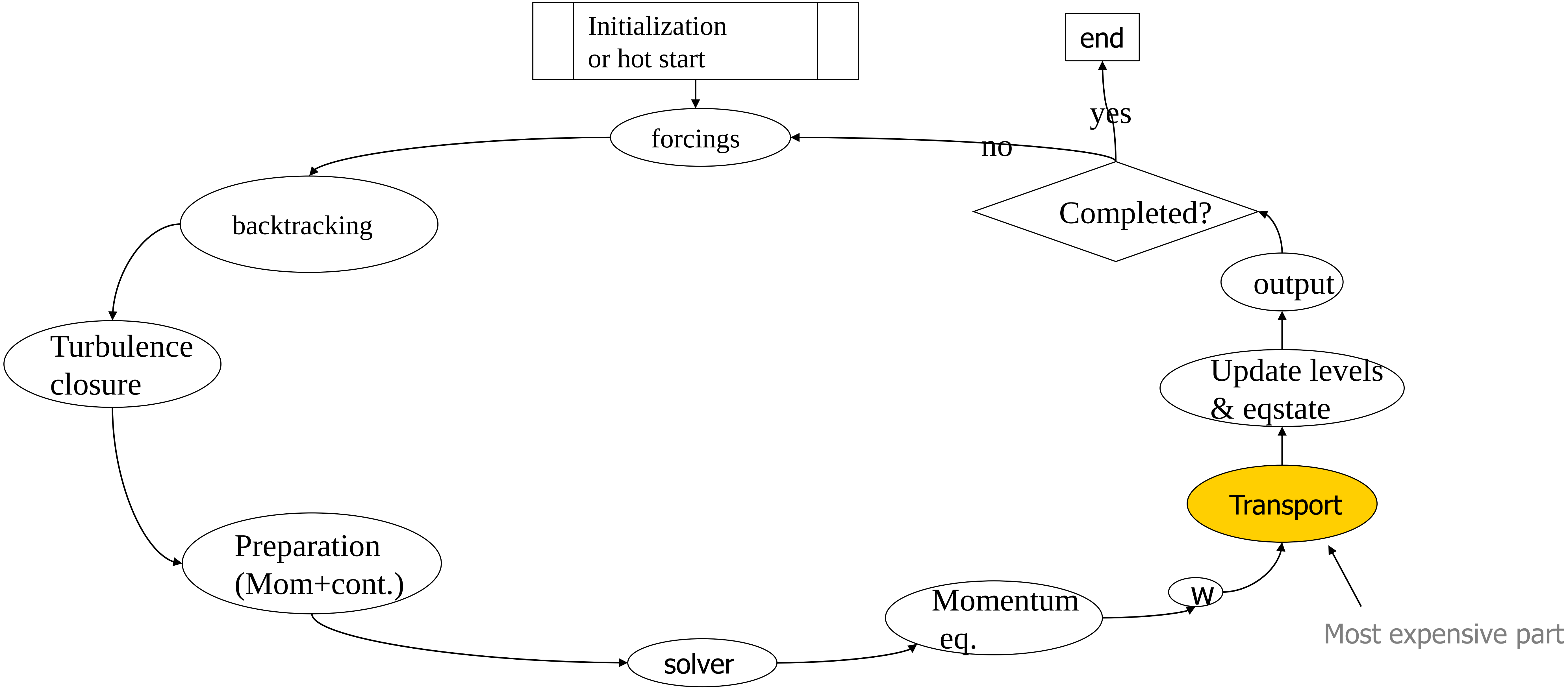
$$\int_{\Omega-\Omega^b}[\ldots]d\Omega = \int_{\Omega-\Omega^b}[\ldots]d\Omega - \theta\Delta t\int_{\Gamma\cup\Gamma^b}\phi_i\widehat{U}^{n+1}d\Gamma$$

$$-(1-\theta)\Delta t\int_{\Gamma\cup\Gamma^b}\phi_i U_n^n d\Gamma - \theta\Delta t\int_{\bar{\Gamma}_v}\phi_i U_n^{n+1}d\bar{\Gamma}_v$$

And for the momentum eq.

- Add Γ^b as a horizontal boundary in backtracking
- At Γ^b and also for all internal sides in Ω^b, the side vel. is imposed from Q – like an open bnd (add to I_3 and I_5)

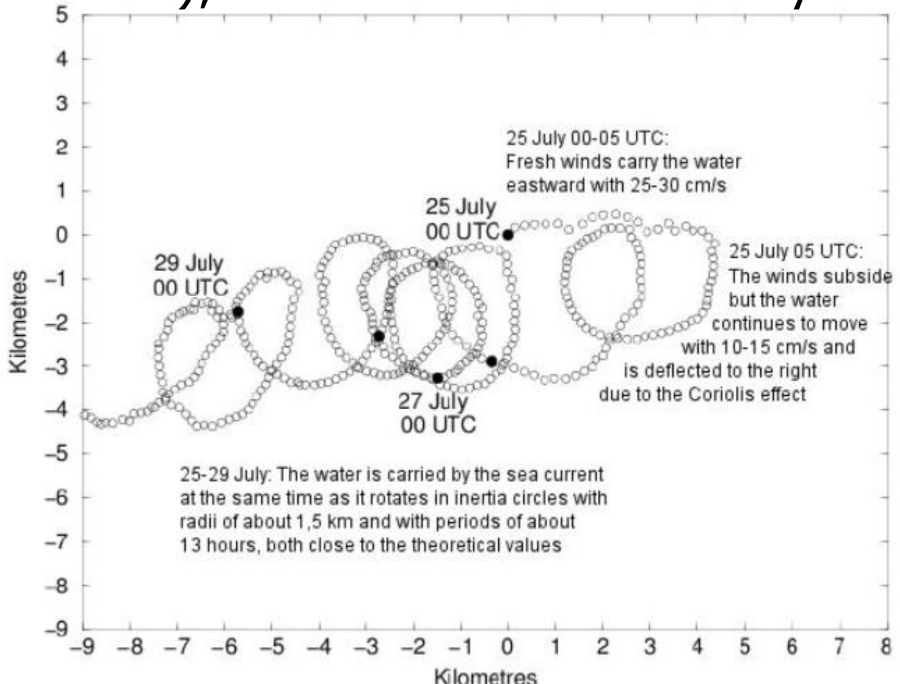
SCHISM flow chart



Numerical stability

- Semi-implicitness circumvents CFL (most severe) (Casulli and Cattani 1994)
- ELM bypasses Courant number condition for advection (actually CFL>0.4)
- Implicit TVD[2] transport scheme along the vertical bypasses Courant number condition in the vertical
- Explicit terms
  - Baroclinicity → internal Courant number restriction (usually not a problem) $\frac{\sqrt{g'h}\Delta t}{\Delta x} < 1$
  - Horizontal viscosity/diffusivity → diffusion number condition (mild) $\frac{\mu \Delta t}{\Delta x^2} \le 0.5$
  - Upwind, TVD[2] transport or WEON[3]: horizontal Courant number condition → **sub-cycling**
  - Coriolis – “inertial modes” in eddying regime (Le Roux 2009; Danilov 2013); needs to be stabilized by viscosity/filter

$$2u_1^{n+1} - u_2^{n+1} - u_3^{n+1} = 0$$



Lateral b.c. and nudging

bctides.in (Types 1–5)

param.in (Nudging/sponge layer)

Variable	Type 1 (*.th)	Type 2:	Type 3	Type 4 (*[23]D.th)	Type 5	Type -1	Type -4, -5 (uv3D.th); nudging	Nudging/sponge layer near boundary
η	**elev.th**: Time history; uniform along bnd	constant	Tidal amp/phases	**elev2D.th.nc**: time- and space-varying along bnd	**elev2D.th.nc:** combination of 3&4	Must =0	✗	✓
S&T, tracers	**[MOD]_?.th** : relax to time history (uniform along bnd) for inflow	Relax to constant for inflow	Relax to i.c. for inflow	**[MOD]_3D.th.nc**: relax to time- and space-varying values along bnd during inflow	✗	✗	✗	✓
u,v	**flux.th**: via discharge **(<0 for inflow!)**	Via discharge **(<0 for inflow!)**	Tides (uniform amp/phase along bnd)	**uv3D.th.nc**: time- and space-varying along bnd (in lon/lat for ics=2)	**uv3D.th.nc:** combination of 3&4 (but tidal amp/phases vary along bnd)	Flather ('0' for η)	Relax to **uv3D.th.nc** (2 separate relaxations for in & outflow)	✓

4H's of SCHISM

- Hybrid FE-FV formulation
- Hybrid horizontal grid
- Hybrid vertical grid
- Hybrid code (openMP-MPI)