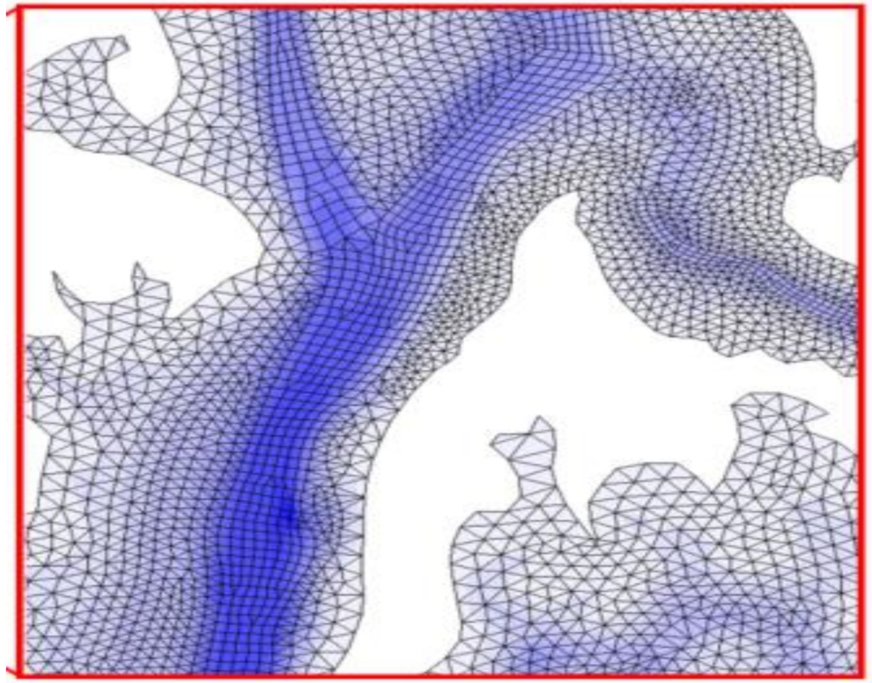
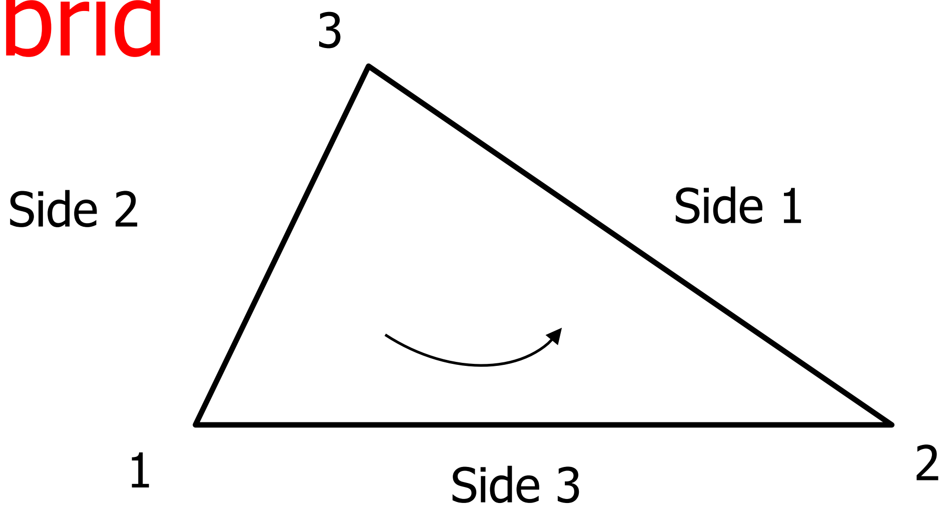
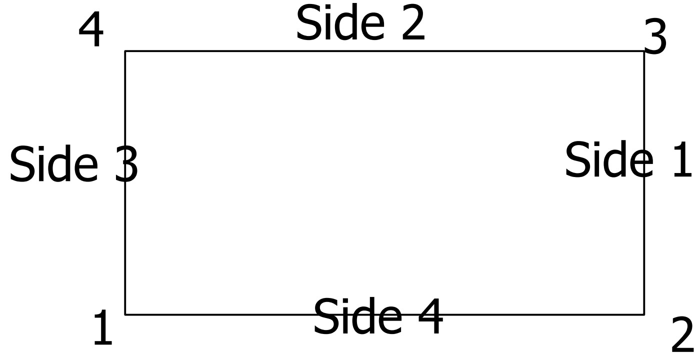
SCHISM numerical formulation

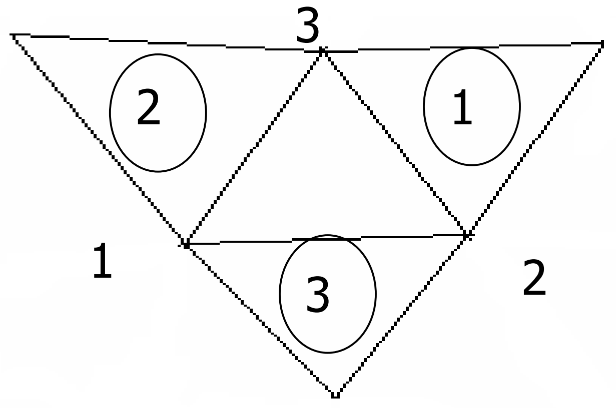
Joseph Zhang

Horizontal grid: hybrid

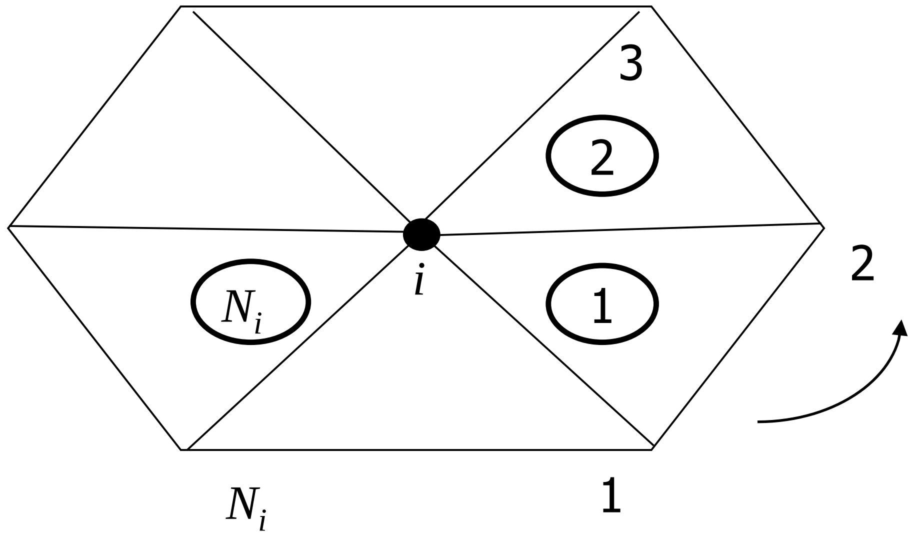


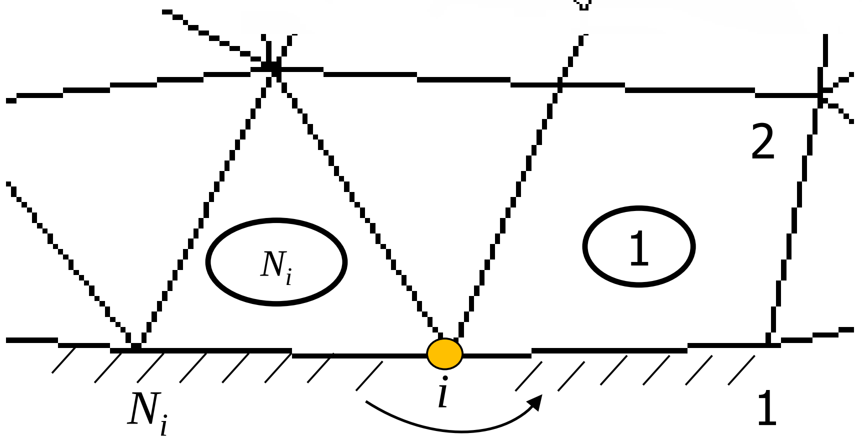


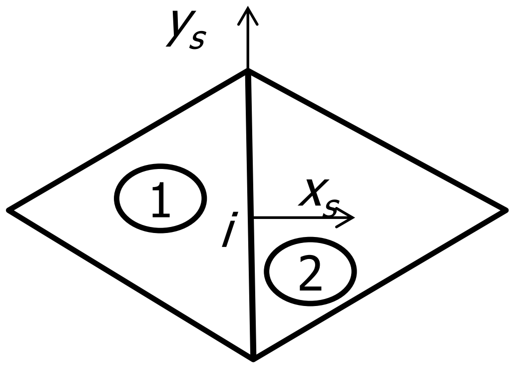




Counter-clockwise convention

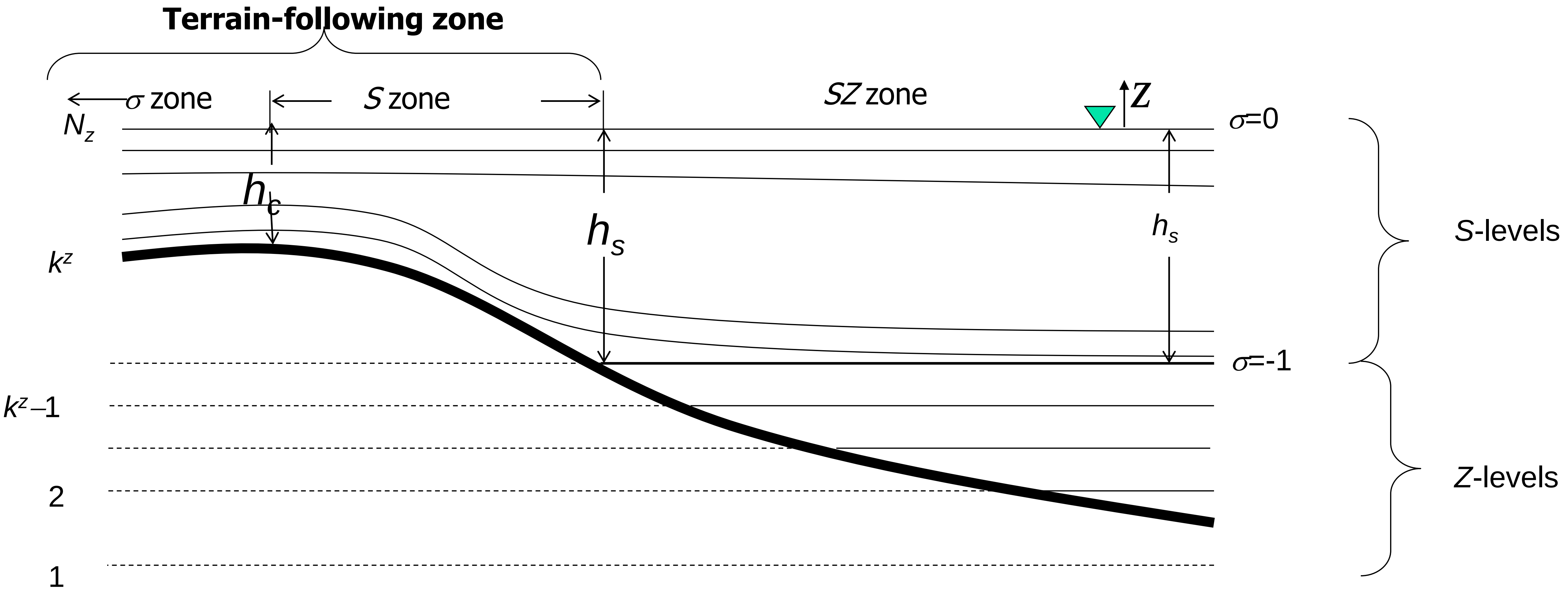






- Internal ‘ball’: starting elem is arbitrary
- Boundary ball: starting elem is unique

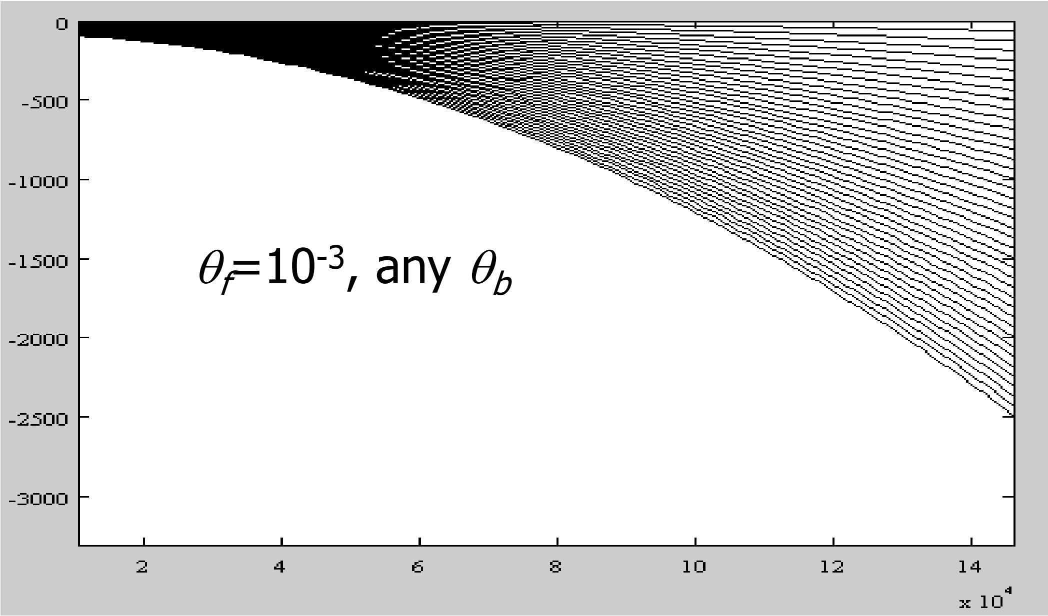
Vertical grid (1): *SZ*

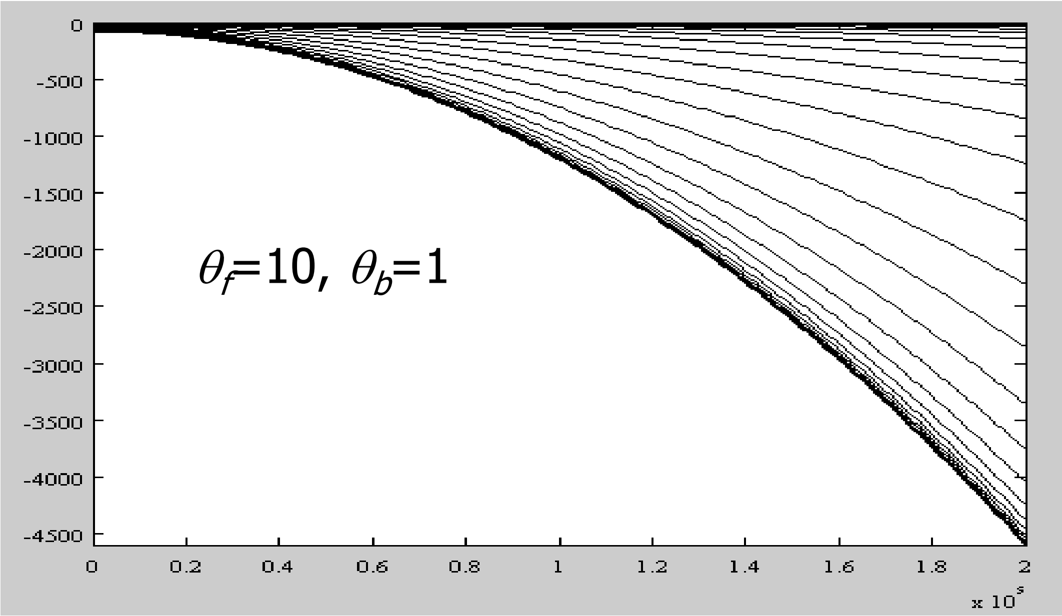


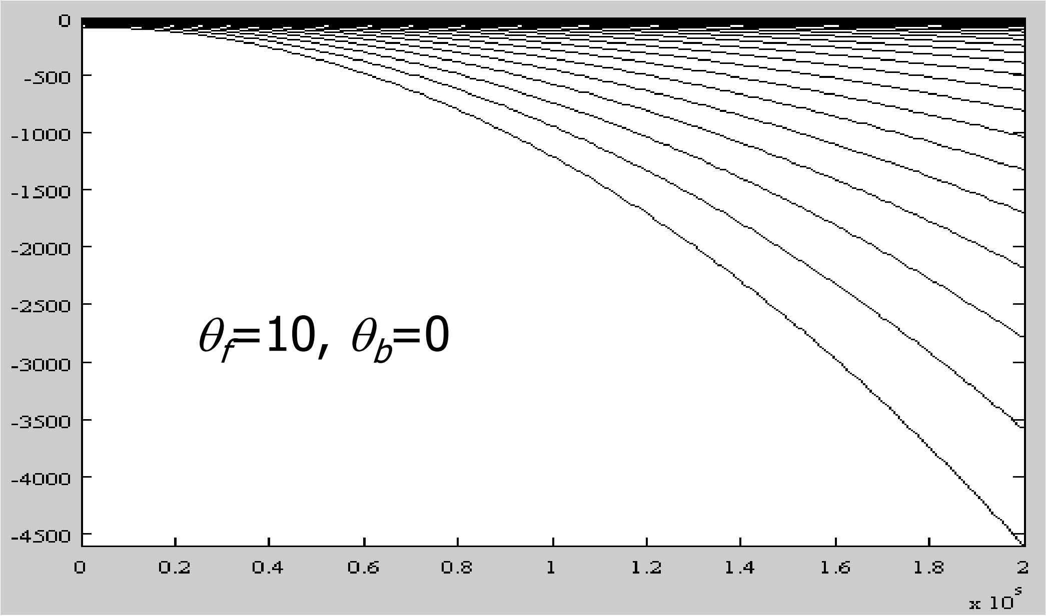
$$\begin{cases} z = \eta(1+\sigma) + h_c\sigma + (\tilde{h} - h_c)C(\sigma) & (-1 \le \sigma \le 0) \\ C(\sigma) = (1-\theta_b)\dfrac{\sinh(\theta_f \sigma)}{\sinh\theta_f} + \theta_b \dfrac{\tanh\left[\theta_f(\sigma+1/2)\right] - \tanh(\theta_f/2)}{2\tanh(\theta_f/2)} & (0 \le \theta_b \le 1;\quad 0 < \theta_f \le 20) \end{cases}$$

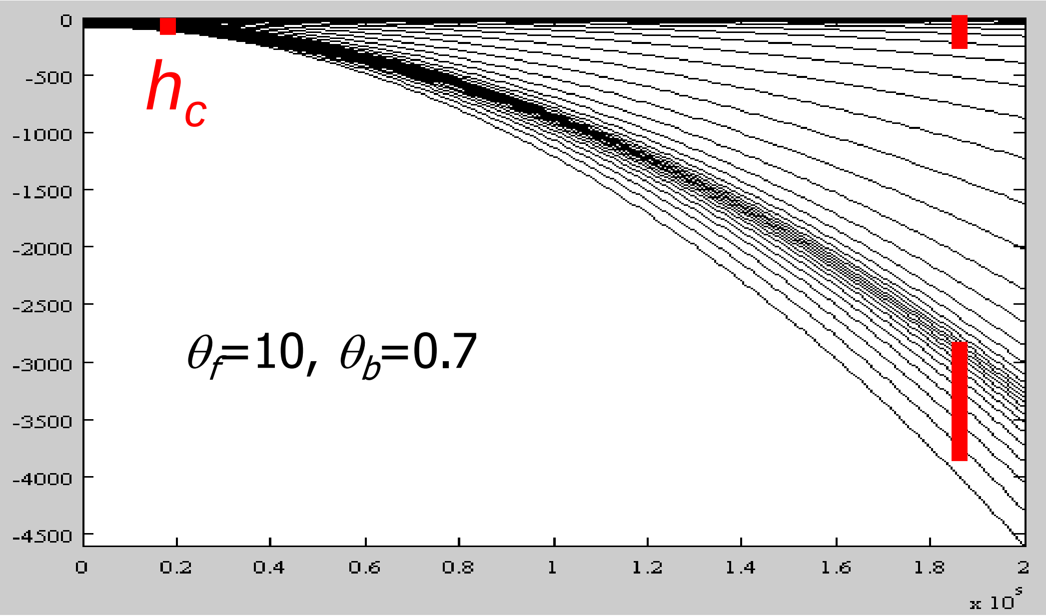
$$\tilde{h} = \min(h, h_s)$$

S-coordinates

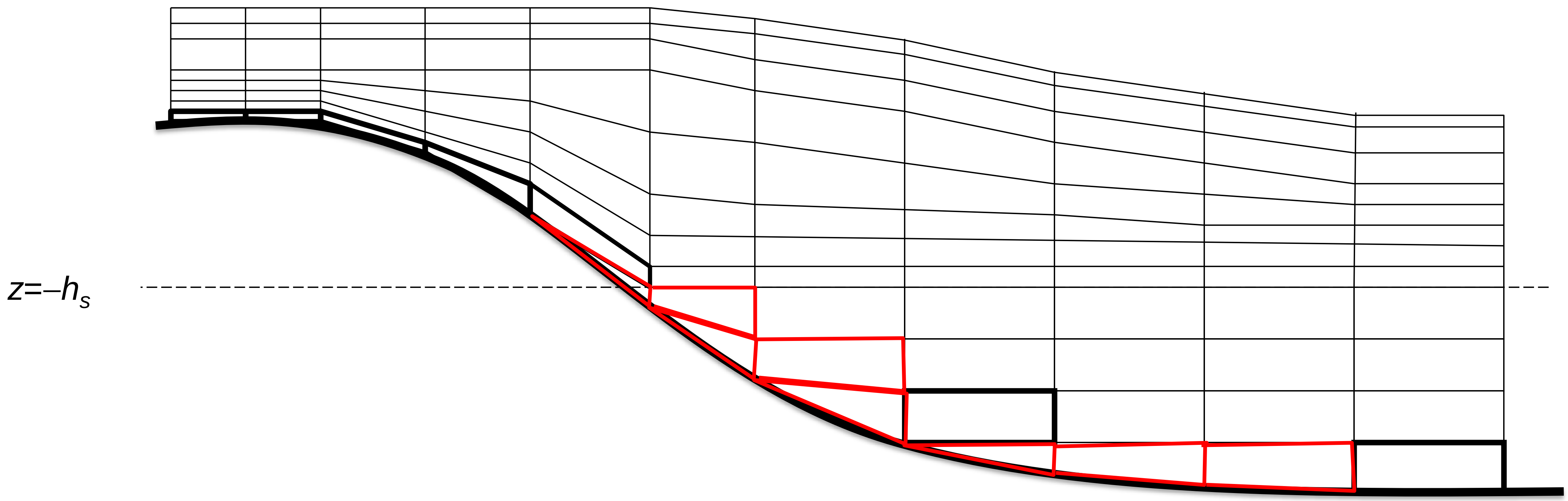






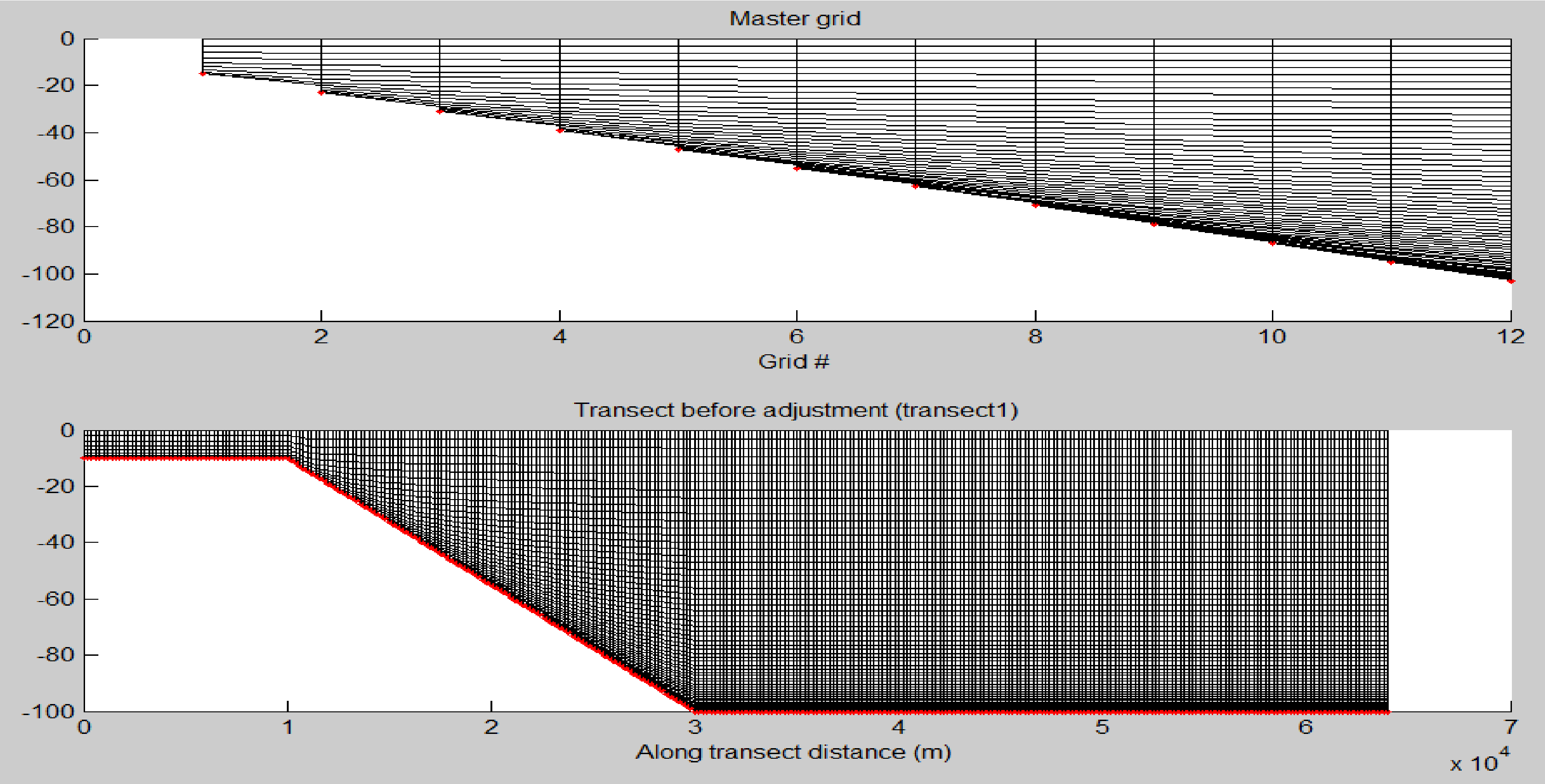


Vertical grid: SZ

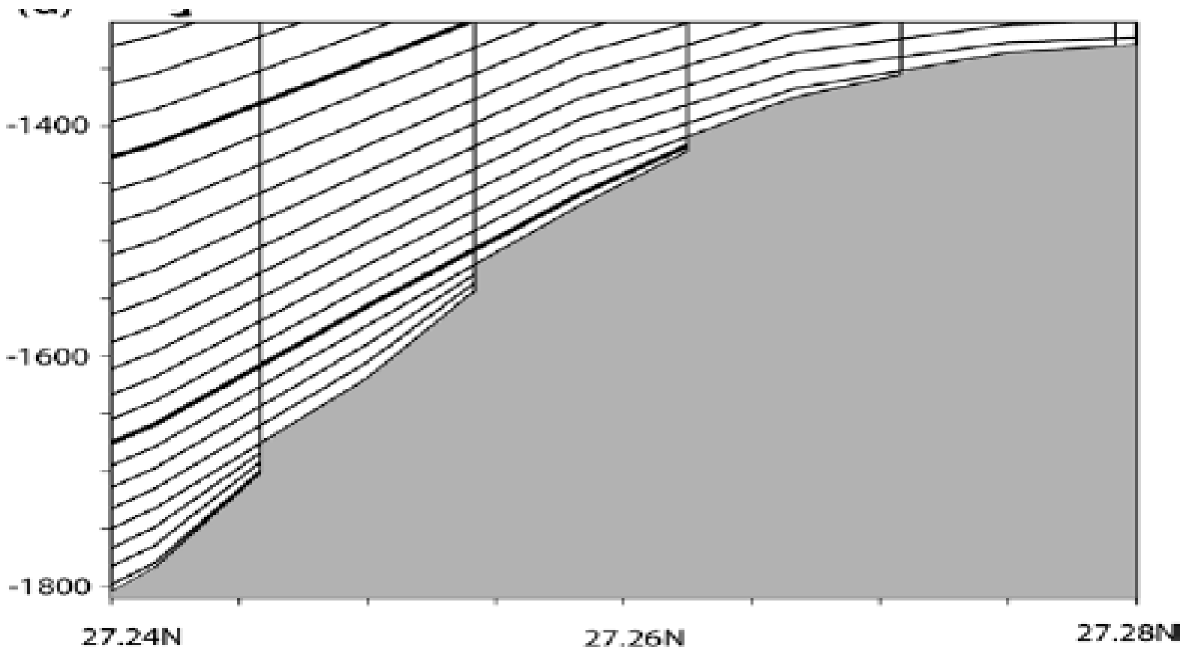


The bottom cells are shaved – no staircases!

Vertical grid (2): LSC^2 via VQS



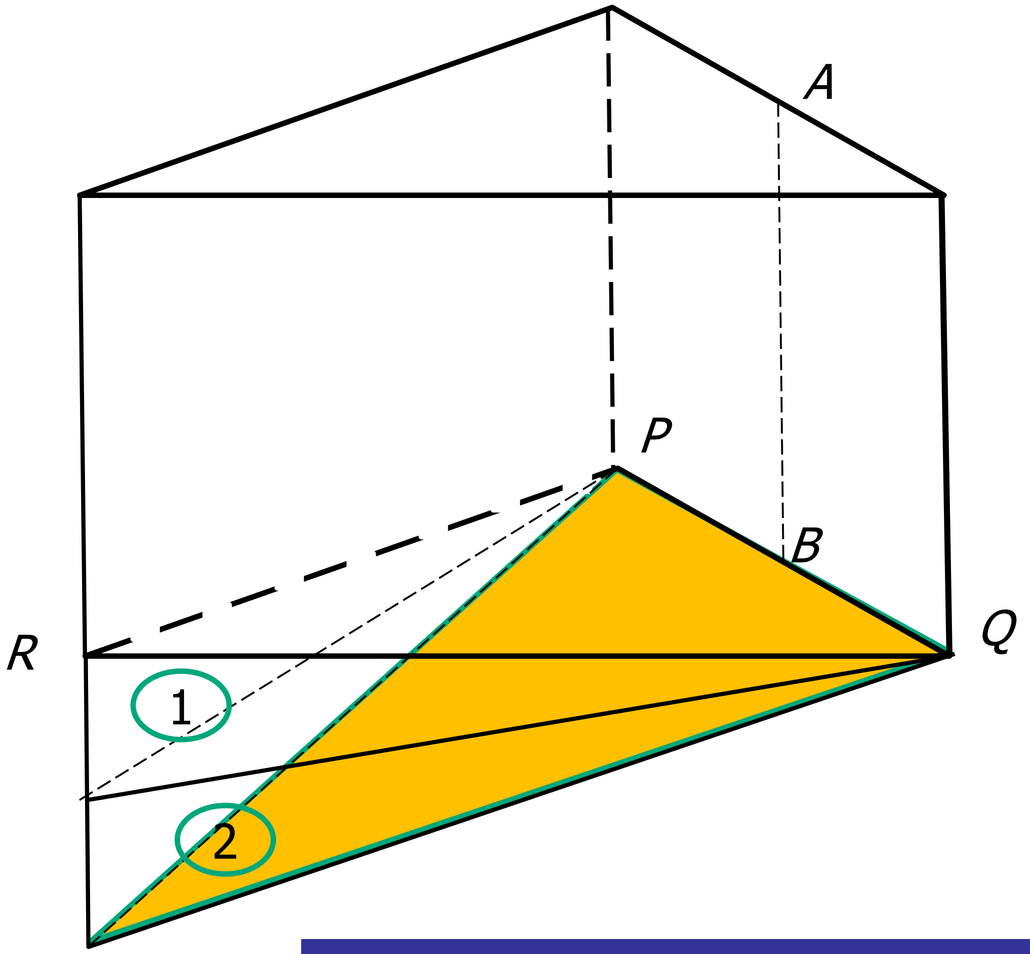
Dukhovskoy et al. (2009)

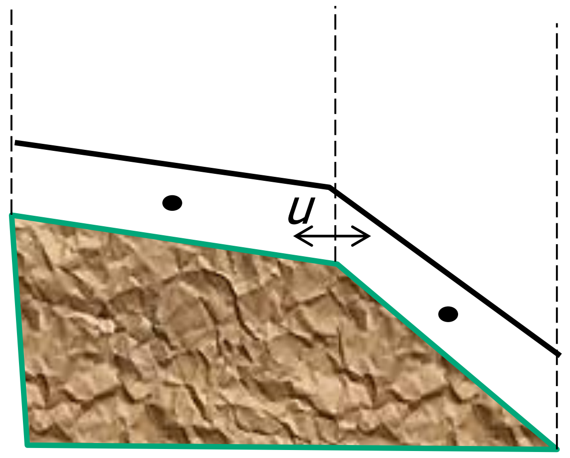


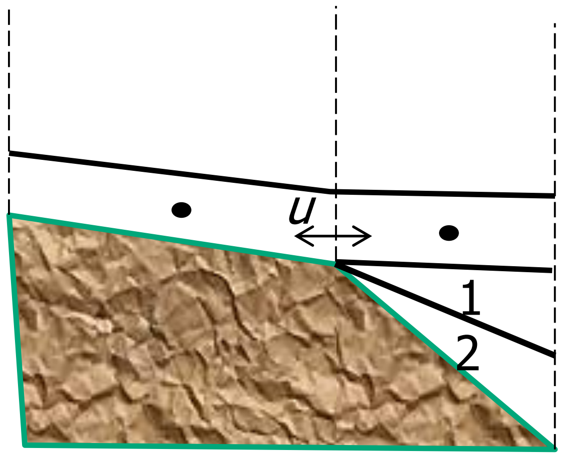
Procedure for VQS

- Different # of levels are used at different depths
- At a given depth, the # of levels is interpolated from the 2 adjacent master grids
- Thin layers near the bottom are masked
- Staircases appear near the bottom (like *Z*), but otherwise terrain following

What to do with the staircases?





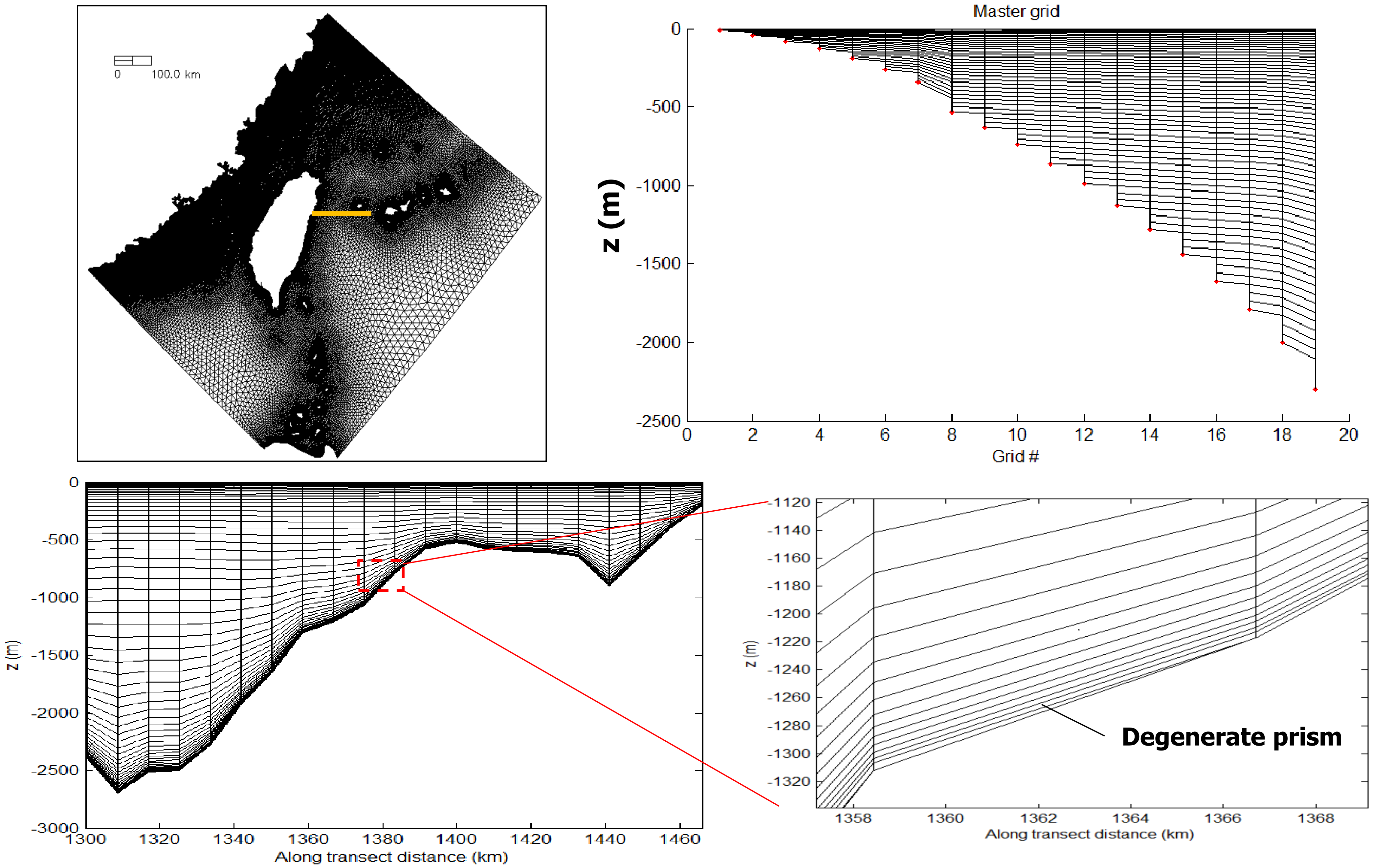


LSC²= Localized Sigma Coordinates with Shaved Cells

(*ivcor* =1 in vgrid.in)

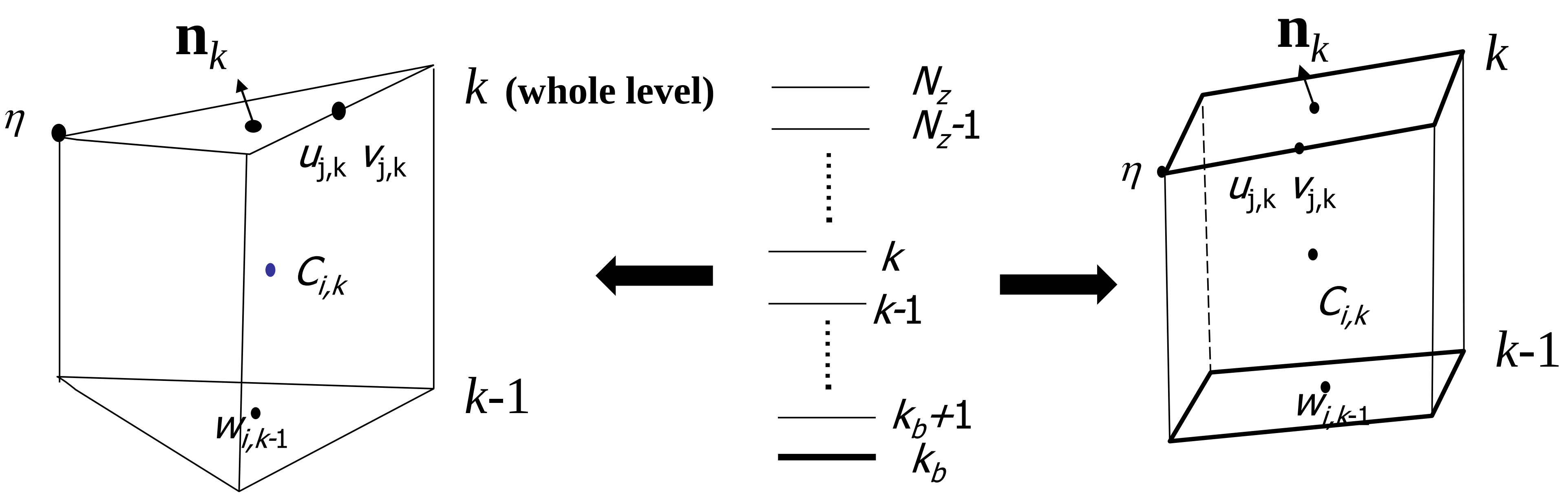
*No masking of thin layers (c/o implicit scheme)

Vertical grid: LSC^2



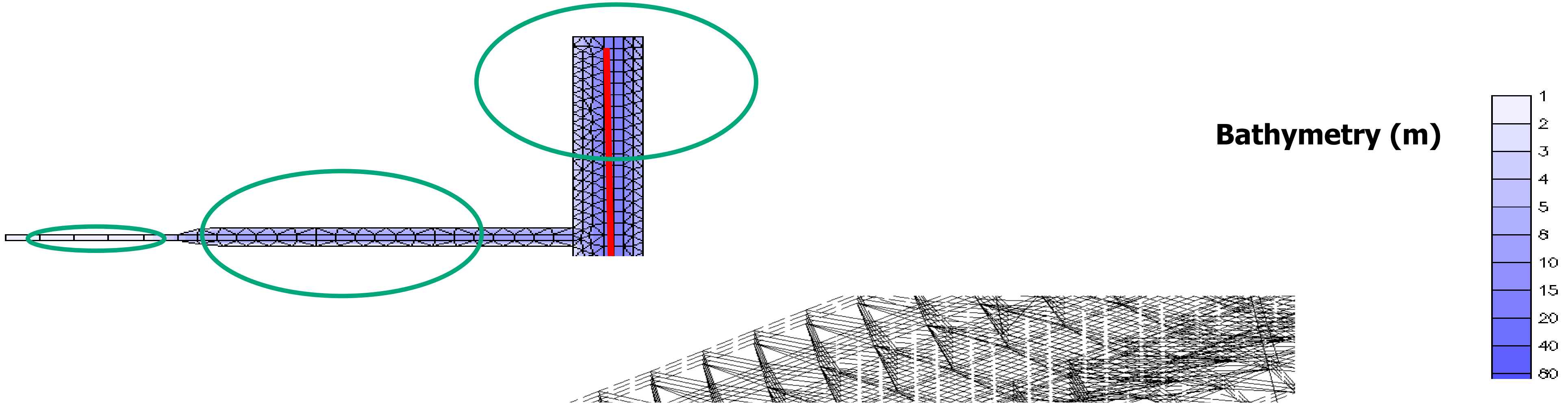
Staggering of variables

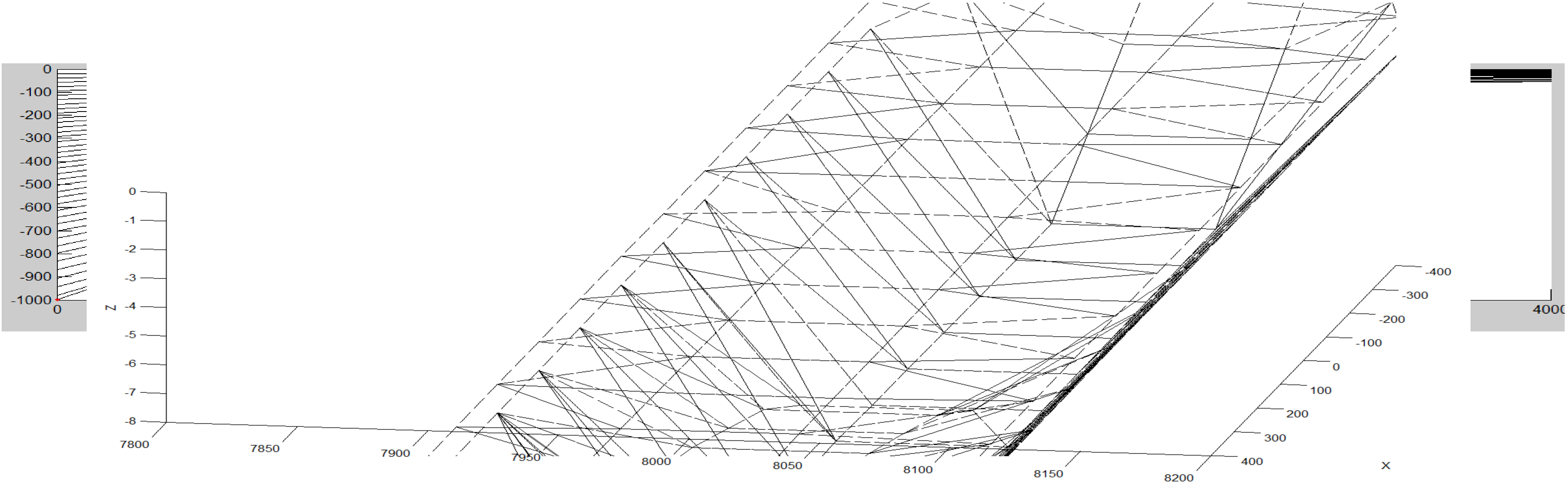
- 3D computational unit: uneven prisms
- **Equations are not transformed into *S*- or *σ*-coordinates, but solved in the original *Z*-space**
- Pressure gradient $\int_z^\eta \nabla \rho d\zeta$
    - *Z*-method with cubic spline (extrapolation on surface/bottom)



Polymorphism

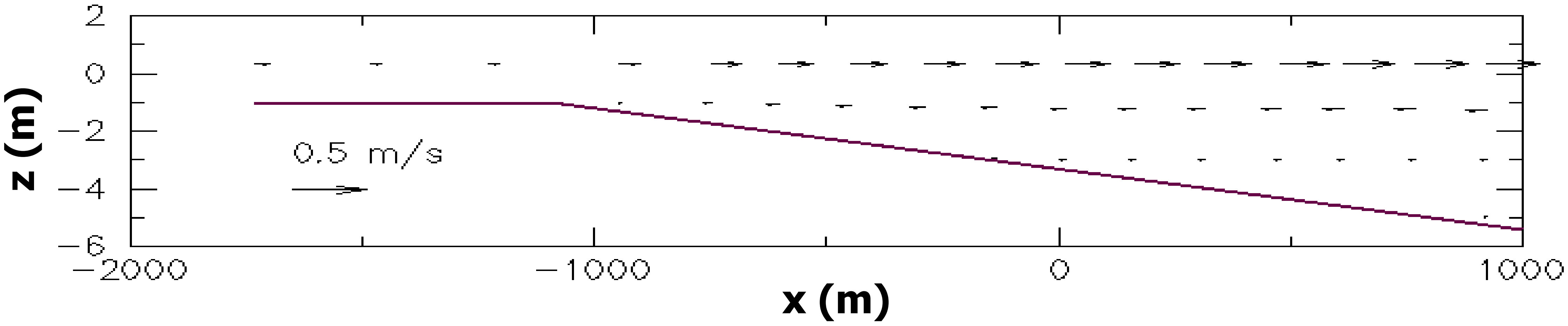
Zhang et al. (2016)

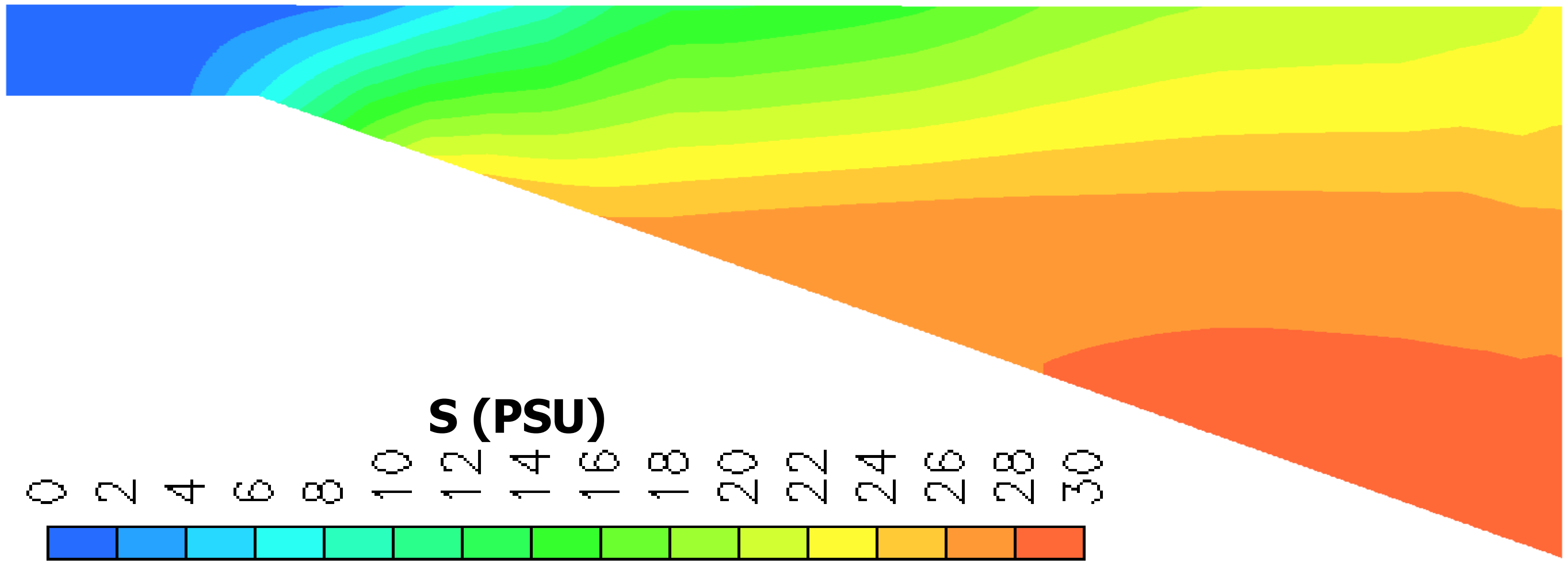




Polymorphism

flood





***Make sure there is no jump in Cd between 2D/3D zones**

Semi-implicit scheme

Continuity

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t}+\theta\nabla\cdot\int_{-h}^{\eta}\mathbf{u}^{n+1}dz+(1-\theta)\nabla\cdot\int_{-h}^{\eta}\mathbf{u}^{n}dz=0$$

Momentum

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{*}}{\Delta t}=\boldsymbol{f}-g\theta\nabla\eta^{n+1}-g(1-\theta)\nabla\eta^{n}+\boldsymbol{m}_{\boldsymbol{z}}^{\boldsymbol{n}+\boldsymbol{1}}-\alpha|\boldsymbol{u}|\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}}L(x,y,z)$$

b.c. (3D)

$$\begin{cases}\nu^{n}\dfrac{\partial\mathbf{u}^{n+1}}{\partial z}=\boldsymbol{\tau}_{w}^{n+1}, & \text{at } z=\eta^{n};\\[2ex] \nu^{n}\dfrac{\partial\mathbf{u}^{n+1}}{\partial z}=\chi^{n}\mathbf{u}_{b}^{n+1}, & \text{at } z=-h,\end{cases}\qquad \chi^{n}=C_{D}\,|\,\mathbf{u}_{b}^{n}\,|$$

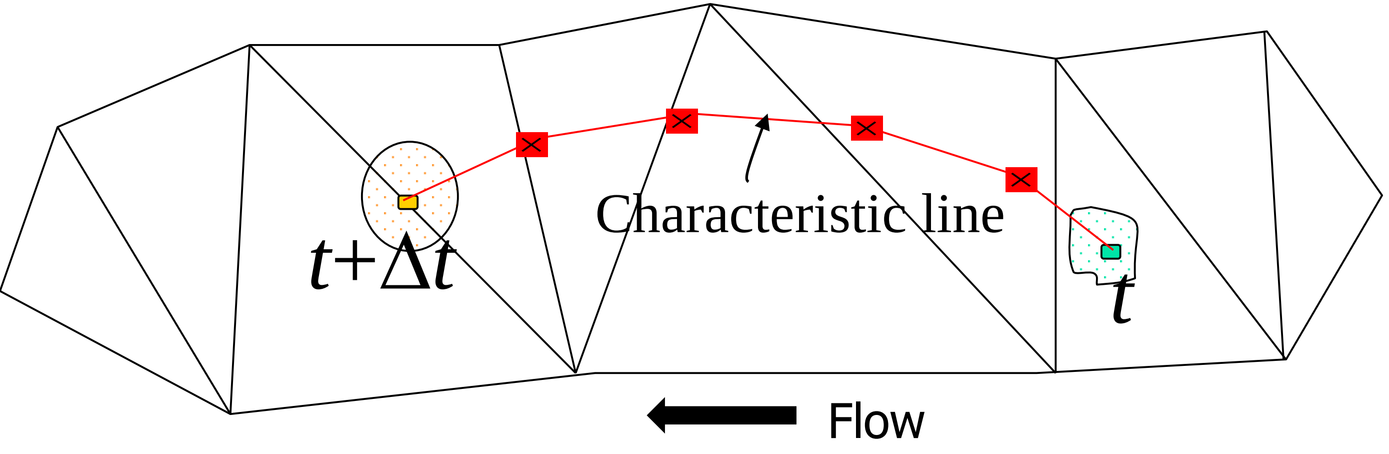
Advection:

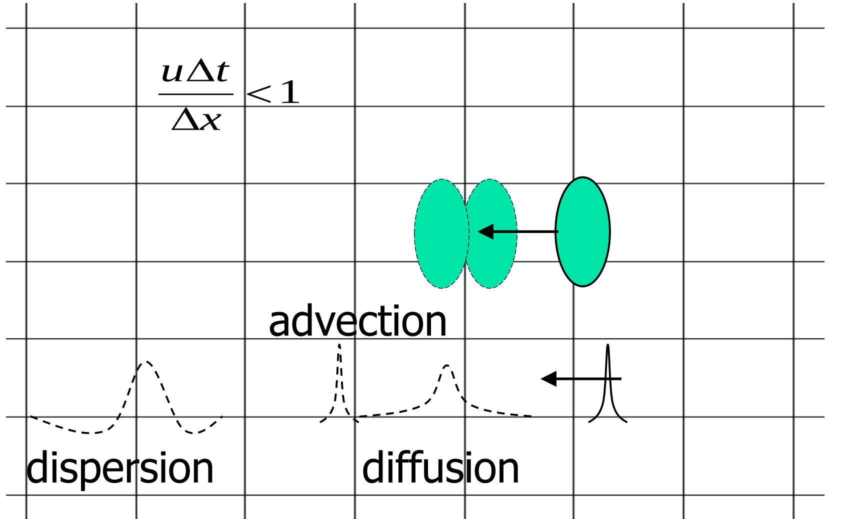
$$\frac{D\mathbf{u}}{Dt}\approx\frac{\mathbf{u}^{n+1}-\mathbf{u}_{*}}{\Delta t};\quad \mathbf{u}_{*}(x,y,z,t+\Delta t,\Delta t)$$

- Implicit treatment of divergence, pressure gradient and SAV form drag terms by-passes most severe stability constraints: $0.5 \leq \theta \leq 1$
- Explicit treatment: Coriolis, baroclinicity, horizontal viscosity, radiation stress…

Eulerian-Lagrangian Method (ELM)

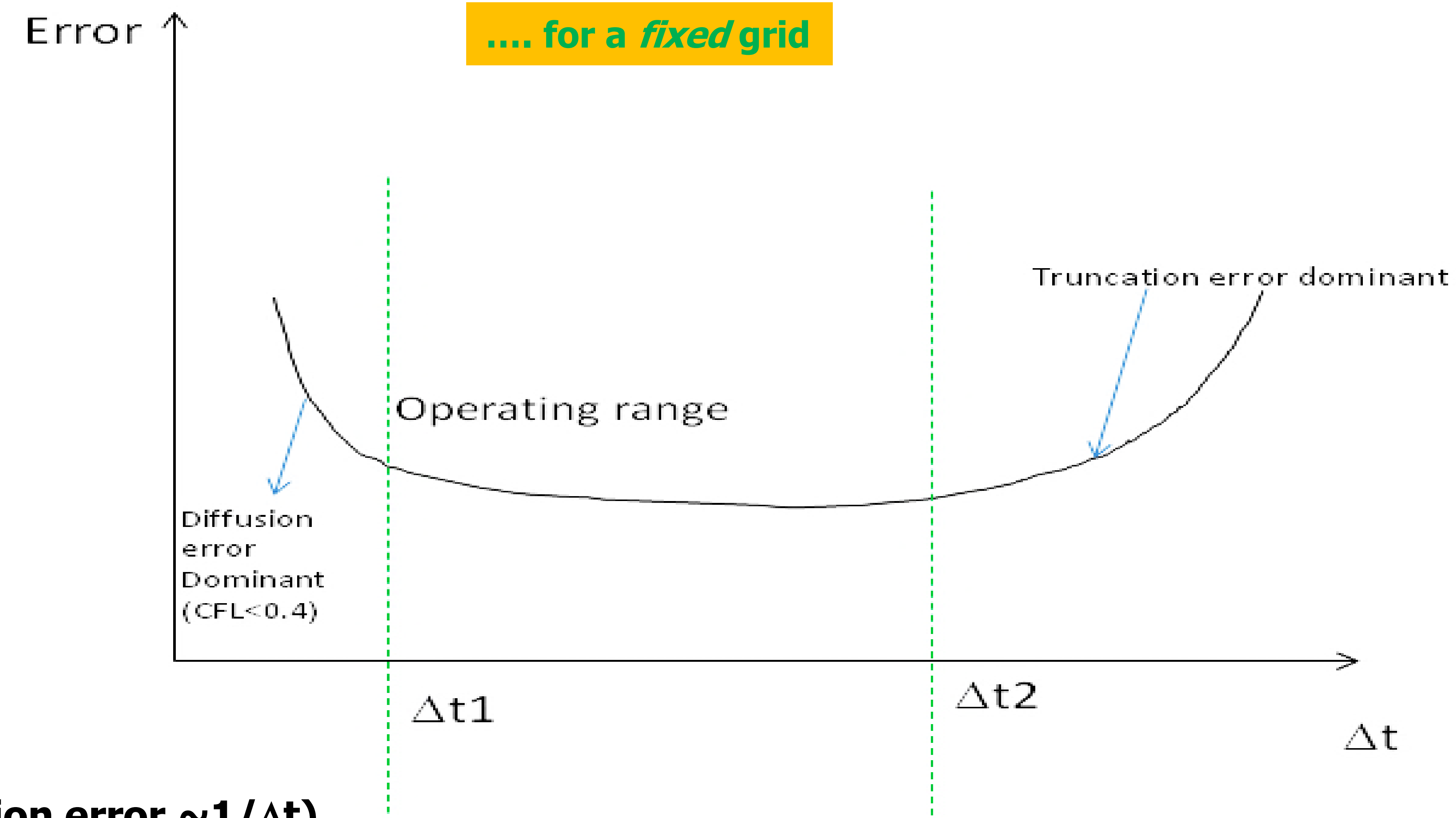
- ELM: takes advantage of both Lagrangian and Eulerian methods
  - Grid is fixed in time, and time step is *not* limited by Courant number condition
  - Advections are evaluated by following a particle that starts at certain point at time t and ends right at a pre-given point at time $t+\Delta t$.
  - The process of finding the starting point of the path (foot of characteristic line) is called *backtracking*, which is done by integrating $d\mathbf{x}/dt=\mathbf{u}_3$ backward in 3D space and time.
  - To better capture the particle movement, the backward integration is often carried out in small sub-time steps
    - Simple backward Euler method
    - 2nd-order R-K method
    - Interpolation-ELM does not conserve; integration ELM does
  - Interpolation: numerical diffusion vs. dispersion





Operating range of SCHISM

$$CFL = \frac{(|\boldsymbol{u}| + \sqrt{gh})\Delta t}{\Delta x}$$



- **ELM prefers larger Δt (truncation error ~1/Δt)**
  - **As a result, SCHISM requires CFL>0.4**
- **Convergence: CFL=const, Δt→0**
- **For barotropic field applications: [100,450] sec**
- **For baroclinic field applications: [100,200] sec (e.g., start with 150 sec)**

ELM with high-order Kriging

- Best linear unbiased estimator for a random function $f(\mathbf{x})$
- "Exact" interpolator
- Works well on unstructured grid (efficient)
- Needs filter (ELAD) to reduce dispersion

$$f(x,y) = \underbrace{\alpha_1 + \alpha_2 x + \alpha_3 y}_{\text{Drift function}} + \underbrace{\sum_{i=1}^{N} \beta_i K(|\mathbf{x} - \mathbf{x}_i|)}_{\text{Fluctuation}}$$

K is called generalized covariance function

Minimizing the variance of the fluctuation we get

Cubic spline

$$K(h) = -h, \quad h^2 \ln h, \quad h^3, \ldots$$

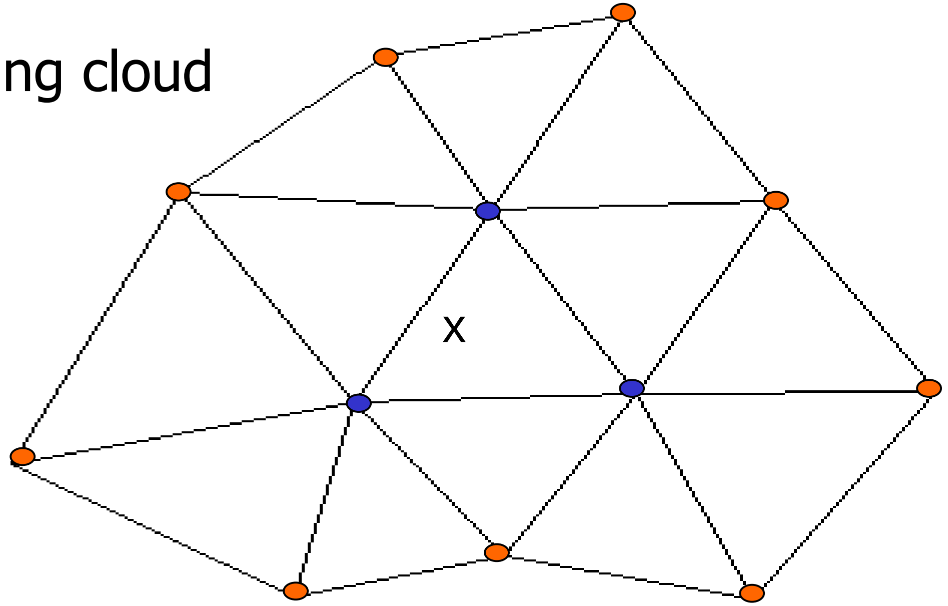
$$\sum_{i=1}^{N} \beta_i = 0$$

$$\sum_{i=1}^{N} \beta_i x_i = 0$$

$$\sum_{i=1}^{N} \beta_i y_i = 0$$

$$f(x_i, y_i) = f_i$$

2-tier Kriging cloud



- The numerical dispersion generated by kriging is reduced by a diffusion ELAD filter (Zhang et al. 2016)

Semi-implicit scheme

Continuity

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t}+\theta\nabla\cdot\int_{-h}^{\eta}\mathbf{u}^{n+1}dz+(1-\theta)\nabla\cdot\int_{-h}^{\eta}\mathbf{u}^{n}dz=0$$

Momentum

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{*}}{\Delta t}=\boldsymbol{f}-g\theta\nabla\eta^{n+1}-g(1-\theta)\nabla\eta^{n}+\boldsymbol{m}_{z}^{n+1}-\alpha|\boldsymbol{u}|\boldsymbol{u}^{n+1}L(x,y,z)$$

b.c.

$$\begin{cases}\nu^{n}\dfrac{\partial\mathbf{u}^{n+1}}{\partial z}=\boldsymbol{\tau}_{w}^{n+1}, & \text{at } z=\eta^{n};\\[2ex] \nu^{n}\dfrac{\partial\mathbf{u}^{n+1}}{\partial z}=\chi^{n}\mathbf{u}_{b}^{n+1}, & \text{at } z=-h,\end{cases}\qquad \chi^{n}=C_{D}\,|\,\mathbf{u}_{b}^{n}\,|$$

$$\frac{D\mathbf{u}}{Dt}\approx\frac{\mathbf{u}^{n+1}-\mathbf{u}_{*}}{\Delta t};\quad \mathbf{u}_{*}(x,y,z,t+\Delta t,\Delta t)$$

- Implicit treatment of divergence and pressure gradient terms by-passes most severe CFL condition : 0.5 ≤θ≤1
- Explicit treatment: Coriolis, baroclinicity, horizontal viscosity, radiation stress…

Galerkin finite-element formulation

A Galerkin weighted residual statement for the continuity equation:

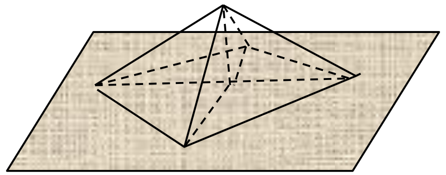
$$\int_{\Omega} \phi_i \frac{\eta^{n+1}-\eta^n}{\Delta t} d\Omega + \theta \left[-\int_{\Omega} \nabla\phi_i \cdot \boldsymbol{U}^{n+1} d\Omega + \int_{\Gamma} \phi_i U_n^{n+1} d\Gamma \right] + (1-\theta) \left[-\int_{\Omega} \nabla\phi_i \cdot \boldsymbol{U}^{n} d\Omega + \int_{\Gamma} \phi_i U_n^{n} d\Gamma \right] = 0,$$

$$(i = 1, \cdots, N_p)$$

ϕ_i: shape/weighting function; **U**: depth-integrated velocity; θ:implicitness factor

Need to eliminate $\mathbf{U}^{n+1}$ to get an equation for η alone. We'll do this with the aid from momentum equation:

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^*}{\Delta t} = \boldsymbol{f} - g\theta\nabla\eta^{n+1} - g(1-\theta)\nabla\eta^n + \boldsymbol{m}_z^{n+1} - \alpha|\mathbf{u}|\boldsymbol{u}^{n+1}L(x,y,z)$$



Shape function (global)

2D case

The unknown velocity can be easily found as:

$$\boldsymbol{U}^{n+1} = \breve{\boldsymbol{G}} - g\theta\Delta t\frac{H^2}{\widetilde{H}}\nabla\eta^{n+1}$$

$$\breve{\boldsymbol{G}} = \frac{H}{\widetilde{H}}[\boldsymbol{U}^* + \Delta t(\boldsymbol{F} + \boldsymbol{\tau}_w - g(1-\theta)H\nabla\eta^n)]$$ (explicit terms)

$$\widetilde{H} = H + (\chi + \alpha|\boldsymbol{u}|H)\Delta t$$

General case

In compact form:

$$\boldsymbol{U}^{n+1} = \boldsymbol{E} - g\theta\breve{H}\Delta t\nabla\eta^{n+1}$$

$$\breve{H} = \begin{cases} \dfrac{H^2}{\widetilde{H}}, & \text{locally 2D} \\ \dfrac{\widehat{H}}{1 + \alpha\overline{|\boldsymbol{u}|}\Delta t}, & \text{locally 3D emergent} \\ \overline{\overline{H}}, & \text{locally 3D submerged} \end{cases}$$

(Note the difference in $\breve{H}$ between 2D/3D)

$$\boldsymbol{E} = \begin{cases} \breve{\boldsymbol{G}}, & \text{2D} \\ \boldsymbol{G}_1, & \text{3D emergent} \\ \boldsymbol{G}_2, & \text{3D submerged} \end{cases}$$

Finite-element formulation (cont'd)

Finally, substituting this eq. back to continuity eq. we get one equation for elevations alone:

$$\underbrace{\int_\Omega \left[\phi_i \eta^{n+1} + g\theta^2 \Delta t^2 \breve{H} \nabla\phi_i \cdot \nabla\eta^{n+1}\right] d\Omega}_{I_1}$$

$$= \underbrace{\int_\Omega [\phi_i \eta^n + \theta\Delta t \nabla\phi_i \cdot \boldsymbol{E} + (1-\theta)\Delta t \nabla\phi_i \cdot \mathbf{U}^n] d\Omega}_{I_4} - \theta\Delta t \underbrace{\int_{\Gamma_v} \phi_i \widehat{U}_n^{n+1} d\Gamma_v}_{I_3} - (1-\theta)\,\Delta t \underbrace{\int_\Gamma \phi_i U_n^n d\Gamma}_{I_5} - \theta\Delta t \underbrace{\int_{\bar{\Gamma}_v} \phi_i U_n^{n+1} d\bar{\Gamma}_v}_{I_6} + I_7$$

sources (→ I_7)

(i=1,…,N_p)

Notes:

- When node i is on boundaries where essential b.c. is imposed, η is known and so no equations are needed, and so I_2 and I_6 need not be evaluated there
- When node i is on boundaries where natural b.c. is imposed, the velocity is known and so the last term on LHS is known → only the first term is truly unknown!
- b.c. is free lunch in FE model (cf. staggering in FV)
- The inherent numerical dispersion in FE nicely balances out the numerical diffusion inherent in the implicit time stepping scheme!

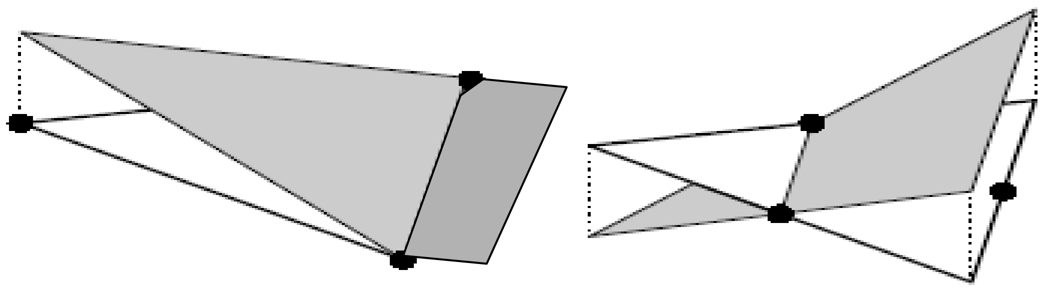
Shape function

- Used to approximate the unknown function
- Although usually used within each individual elements, shape function is global not local!

$$u = \sum_{i=1}^{N} u_i \phi_i, \quad \phi_i(x_j) = \delta_{ij}$$

- Must be sufficiently smooth to allow integration by part in weak formulation
- Mapping between local and global coordinates
- Assembly of global matrix

$$\int L\left(\sum_{i=1}^{N} u_i \phi_i\right)\phi_j d\Omega = \sum_{m=1}^{M} \int_{\Omega_m} L\left(\sum_{i=1}^{N} u_i \phi_i\right)\phi_j d\Omega_m = 0, \quad j = 1, \ldots, N$$

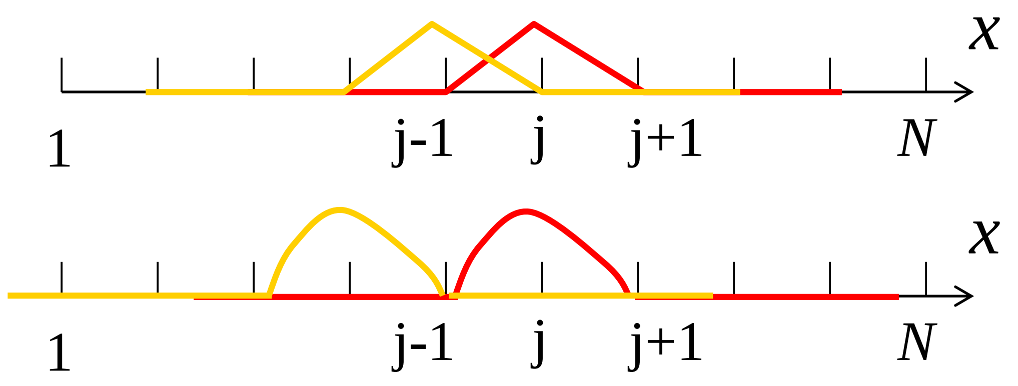


Conformal

(Elevation, velocity)

Non-conformal

(indvel=0 -> less dissipation and needs further stabilization (MB* schemes)

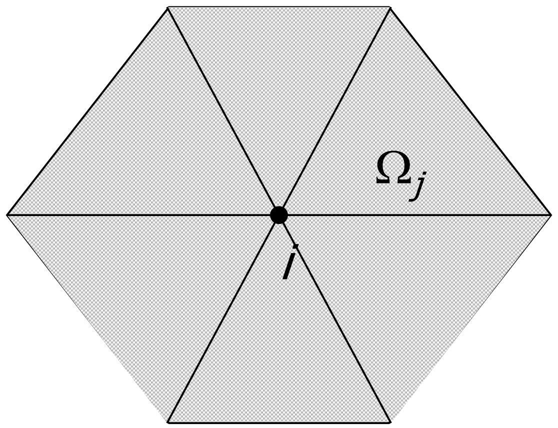


Matrices: I_1

$$I_1 = \sum_{j=1}^{N_i} \int_{\Omega_j} (\hat{\phi}_i \eta^{n+1} + g\theta^2 \Delta t^2 \breve{H} \nabla \hat{\phi}_{i'} \cdot \nabla \eta^{n+1}) d\Omega_j$$

$$= \sum_{j=1}^{N_i} \sum_{l=1}^{i34(j)} \eta_{j,l}^{n+1} \left[\frac{1+\delta_{i',l}}{12} A_j + \frac{g\theta^2 \Delta t}{4A_j} \bar{\breve{H}} \vec{i'} \cdot \vec{l} \right]$$

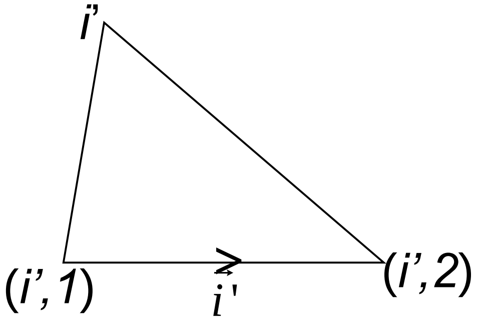
i' is local index of node *i* in Ω_j.



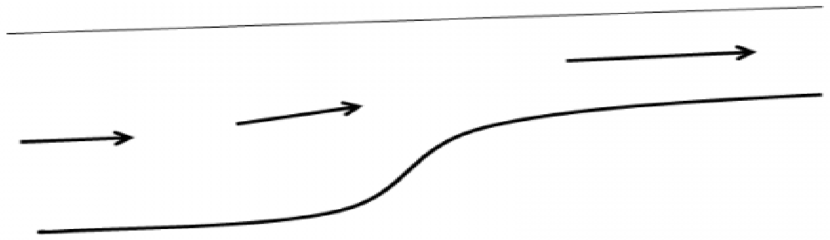
$\vec{i'} \cdot \vec{l}$ reaches its max when *i'*=*l*, and so does I_1.

$$\sum_{i'=1}^{3} \vec{i'} = 0$$

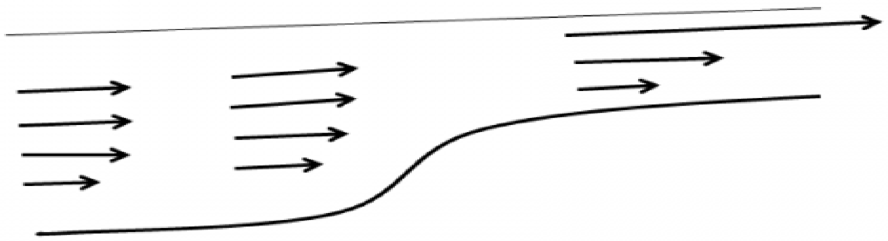
so diagonal is dominant **if the averaged friction-reduced depth ≥0** (can be relaxed)



Mass matrix is positive definite and symmetric! JCG or PETSc solvers work fine



(a) 2D model velocity

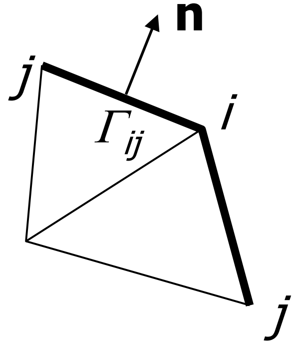


(b) 3D model velocity

Matrices: I_3 and I_5

$$I_3 = \int_{\Gamma_v} \phi_i \hat{U}_n^{n+1} d\Gamma_v = \sum_j \int_{\Gamma_{ij}} \phi_i \hat{U}_n^{n+1} d\Gamma_{ij}$$

$$= \sum_j \frac{L_{ij}}{4} \sum_{k=k_{bs}}^{N_z-1} \Delta z_{ij,k+1} \left[(\hat{u}_\mathrm{n}^{n+1})_{ij,k+1} + (\hat{u}_\mathrm{n}^{n+1})_{ij,k} \right]$$



Flather b.c. (overbar denotes mean)

$$\hat{u}_\mathrm{n}^{n+1} - \overline{u}_\mathrm{n} = \sqrt{g/h}(\eta^{n+1} - \overline{\eta})$$

$$I_3 = \sum_j L_{ij} \left\{ \frac{(\overline{U}_\mathrm{n})_{ij}}{2} + \frac{\sqrt{gh_{ij}}}{6} \left[2(\eta_i^{n+1} - \overline{\eta}_i) + (\eta_j^{n+1} - \overline{\eta}_j) \right] \right\}$$

Similarly

$$I_5 = \sum_j \int_{\Gamma_{ij}} \hat{\phi}_i \hat{U}_n^n d\Gamma_{ij} = \sum_j \frac{L_{ij}}{4} \sum_{k=k_{bs}}^{N_z-1} \Delta z_{ij,k+1} \left[(\hat{u}_\mathrm{n}^n)_{ij,k+1} + (\hat{u}_\mathrm{n}^n)_{ij,k} \right]$$

Matrices: I_7

Related to source/sink (including precipitation), defined at some elements

$$I_7 = \frac{\Delta t}{3} \sum_{j \in B_i \cap (\cup \Omega_m)} s_j(t), \ (i = 1, \ldots, N_p)$$

B_i: ball of node *i*
$U\Omega_m$: elements that have source/sink
S_j: volume discharge rate [m^3/s]
* For better conservation the source/sink are placed at the bottom prism by default, but can be changed
* You can inject sources/sinks anywhere you want!

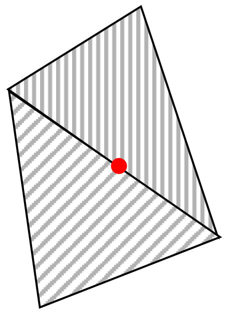
Matrices: I_4

Has the most complex form

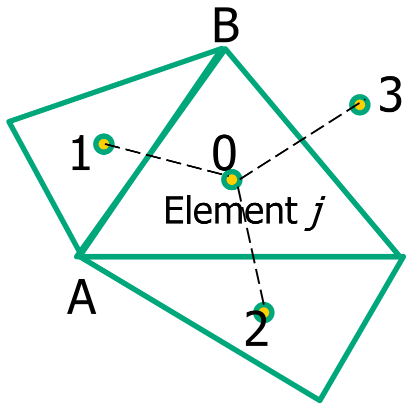
$$I_3 = \int_\Omega \left[\phi_i \eta^n + (1-\theta)\Delta t \nabla \phi_i \cdot \mathbf{U}^n + \theta \Delta t \nabla \phi_i \cdot \boldsymbol{E} \right] d\Omega = \sum_j^{N_i} \left[\frac{A_j}{12} \sum_{l=1}^{i34(j)} \eta^n_{j,l} (1 + \delta_{i',l}) + (1-\theta)\Delta t A_j \nabla \hat{\phi}_{i'} \cdot \bar{\mathbf{U}}^n_j + \theta \Delta t \nabla \hat{\phi}_i \cdot \hat{\mathbf{G}}'_j \right]$$

Most complex part of ***E*** is the baroclinicity

$$\mathbf{f}_c = -\frac{g}{\rho_0} \int_z^\eta \nabla \rho \, d\zeta$$ at elements/sides



Baroclinicity: *z*-method



- Interpolate density along horizontal planes
  - Advantage: alleviate pressure gradient errors (Fortunato and Baptista 1996)
  - Disadvantage: near surface or bottom
- Density is defined at prism centers (half levels)
- Re-construction method: compute directional derivatives along two direction at node *i* and then convert them back to *x,y*
- Density gradient calculated at prism centers first (for continuity eq.); the values at face centers are averaged from those at prism centers (for momentum eq.)
  - constant extrapolation (shallow) is used near bottom (under resolution)
  - Cubic spline is used in all interpolations of ρ
  - Mean density profile $\bar{\rho}(z)$ can be optionally removed
- 3 or 4 eqs for the density gradient vs. 2 unknowns – averaging needed for 3 pairs; e.g.

$$\begin{cases} (\rho_x^{'})_{i,k}(x_1 - x_0) + (\rho_y^{'})_{i,k}(y_1 - y_0) = \rho_1^{'} - \rho_0^{'} \\ (\rho_x^{'})_{i,k}(x_2 - x_0) + (\rho_y^{'})_{i,k}(y_2 - y_0) = \rho_2^{'} - \rho_0^{'} \end{cases}$$

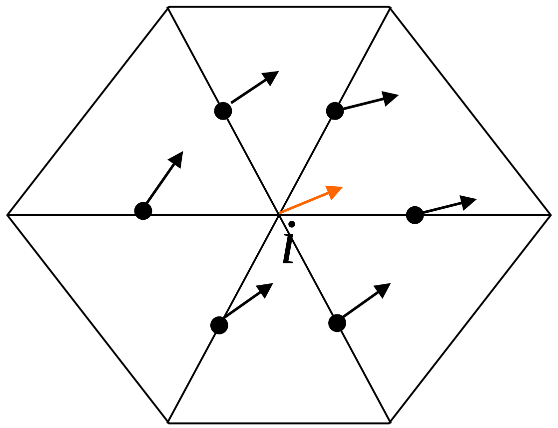
  - Degenerate cases: when the two vectors are co-linear (discard)
  - Vertical integration using trapzoidal rule

Momentum equation: 2D case

The depth-averaged velocity can be directly solved once unknown elevations are found:

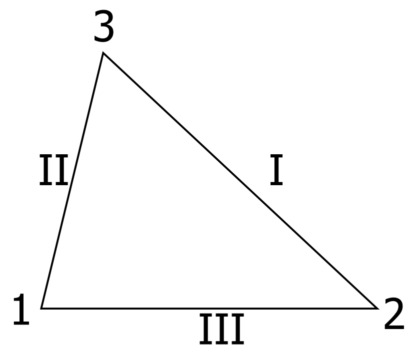
$$\boldsymbol{u}^{\boldsymbol{n+1}} = \frac{\breve{H}}{H}\left[\boldsymbol{u}^{*} + (\boldsymbol{f} + \boldsymbol{\tau}_{\boldsymbol{w}}/H)\Delta t - g\theta\nabla\eta^{n+1} - g(1-\theta)\nabla\eta^{n}\right]$$

Velocity at nodes



Needed for ELM (interpolation), and can serve as a way to further stabilize the mom solver by adding some inherent numerical diffusion

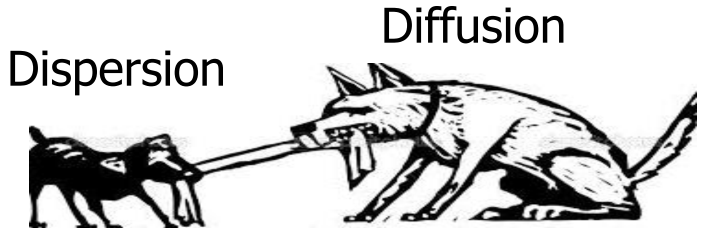
- Method 1: inverse-distance averaging around ball (*indvel*=1 or MA schemes) – larger dissipation; no further stabilization needed
- Method 2: use linear shape function (conformal) (*indvel*=0 or MB schemes)



$$\mathbf{u}_1 = \mathbf{u}_{II} + \mathbf{u}_{III} - \mathbf{u}_I$$

  - Discontinuous across elements; averaging to get the final value at node
  - Generally requires velocity b.c. at open boundary (mostly for incoming info)
  - indvel=0 generally gives less dissipation but needs further stabilization in the form of viscosity or Shapiro filter for stabilization (important for eddying regime)

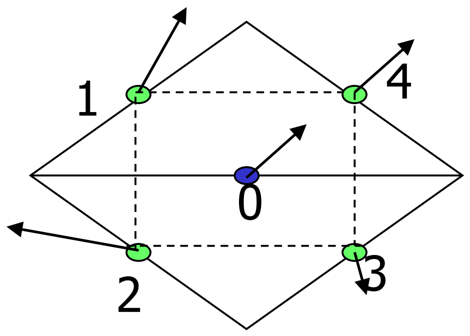
- An ideal numerical algorithm should strike a balance between dispersion and diffusion (dissipation). Neither dispersion nor dissipation is good!



Eternal struggle between diffusion and dispersion

Shaprio filter

A simple filter to reduce sub-grid scale (unphysical) oscillations while leaving the physical signals largely intact (α=0.5 optimal)



$$\hat{\mathbf{u}}_0 = \mathbf{u}_0 + \frac{\alpha}{4}\left(\sum_{i=1}^{4}\mathbf{u}_i - 4\mathbf{u}_0\right), \quad (0 \le \alpha \le 0.5)$$

- No filtering for boundary sides – so need b.c. there especially for incoming flow
- Equivalent to Laplacian viscosity

Horizontal viscosity

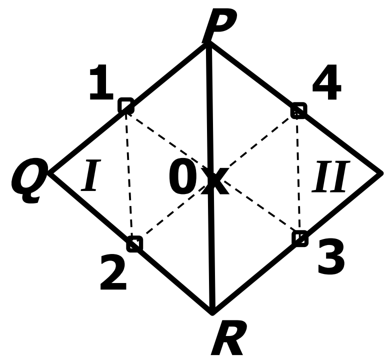
Important for *eddying regime* for adding proper dissipation

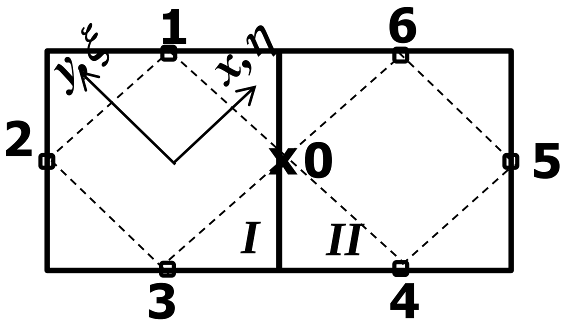
$$\nabla\cdot(\mu\nabla u)\big|_0 \cong \frac{\mu_0}{A_I + A_{II}}\oint_\Gamma \frac{\partial u}{\partial n}d\Gamma$$

Assuming uniformity

(like Shapiro filter)

$$\nabla\cdot(\mu\nabla u)\big|_0 = \frac{\mu_0}{\sqrt{3}A_I}(u_1+u_2+u_3+u_4-4u_0)$$

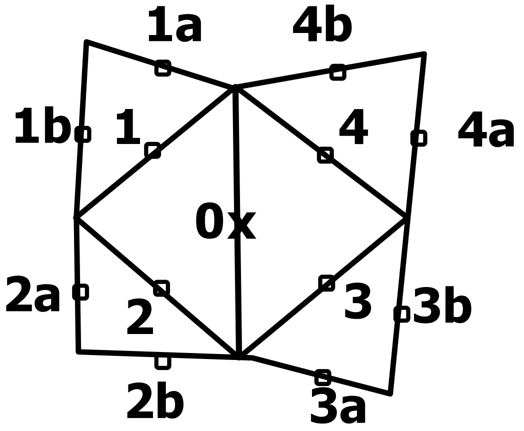




Bi-harmonic viscosity (ideal in eddying regime)

$$-\lambda\nabla^4 u\big|_0 = -\lambda\gamma_3(\nabla^2u_1+\nabla^2u_2+\nabla^2u_3+\nabla^2u_3-4\nabla^2u_0) =$$

$$\frac{\gamma_2}{\Delta t}\left[7(u_1+u_2+u_3+u_4)-u_{1a}-u_{1b}-u_{2a}-u_{2b}-u_{3a}-u_{3b}-u_{4a}-u_{4b}-20u_0\right]$$



(Zhang et al. 2016)

Vertical velocity

- Vertical velocity using Finite Volume

$$\hat{S}_{k+1}(\overline{u}_{k+1}^{n+1} n_{k+1}^x + \overline{v}_{k+1}^{n+1} n_{k+1}^y + w_{i,k+1}^{n+1} n_{k+1}^z) - \hat{S}_k(\overline{u}_k^{n+1} n_k^x + \overline{v}_k^{n+1} n_k^y + w_{i,k}^{n+1} n_k^z) +$$

$$\sum_{m=1}^{3} \hat{P}_{js(i,m)} s_{i,m} (\hat{q}_{js(i,m),k}^{n+1} + \hat{q}_{js(i,m),k+1}^{n+1})/2 = 0, \quad (k = k_b, \ldots, N_z - 1)$$

$$\hat{q}_{j,k} = \mathbf{u}_{j,k} \cdot \mathbf{n}_j$$

i34(j)

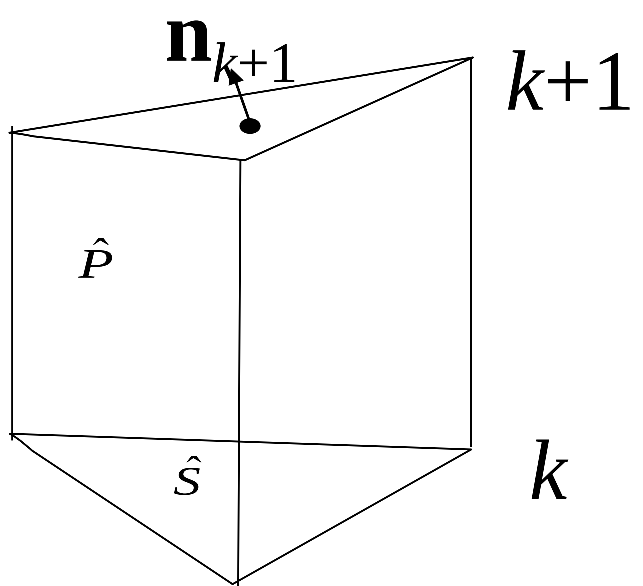
$$w_{i,k+1}^{n+1} = \frac{1}{n_{k+1}^z \hat{S}_{k+1}} \left[-\sum_{m=1}^{3} \hat{P}_{js(i,m)} s_{i,m} (\hat{q}_{js(i,m),k}^{n+1} + \hat{q}_{js(i,m),k+1}^{n+1})/2 - \hat{S}_{k+1}(\overline{u}_{k+1}^{n+1} n_{k+1}^x + \overline{v}_{k+1}^{n+1} n_{k+1}^y) - \hat{S}_k(\overline{u}_k^{n+1} n_k^x + \overline{v}_k^{n+1} n_k^y + w_{i,k}^{n+1} n_k^z) \right],$$

$$(k = k_b, \ldots, N_z - 1)$$

In compact form:

$$\sum_{j \in S^+} |Q_j| = \sum_{j \in S^-} |Q_j|$$

This is the foundation of mass conservation and monotonicity in transport solvers



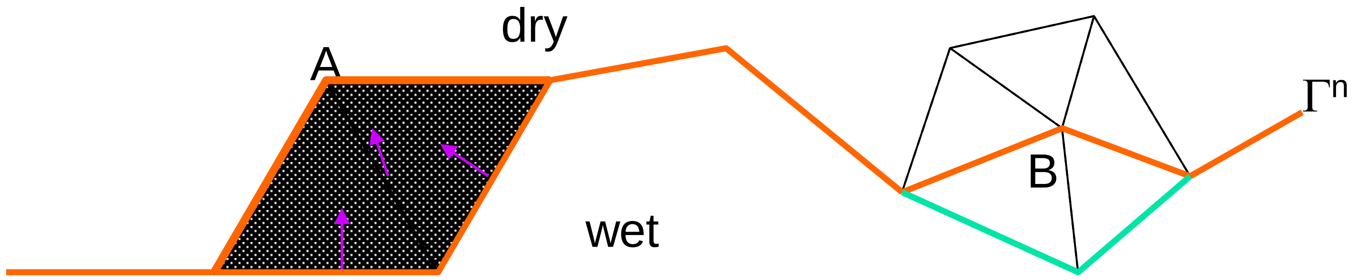
Bottom b.c.

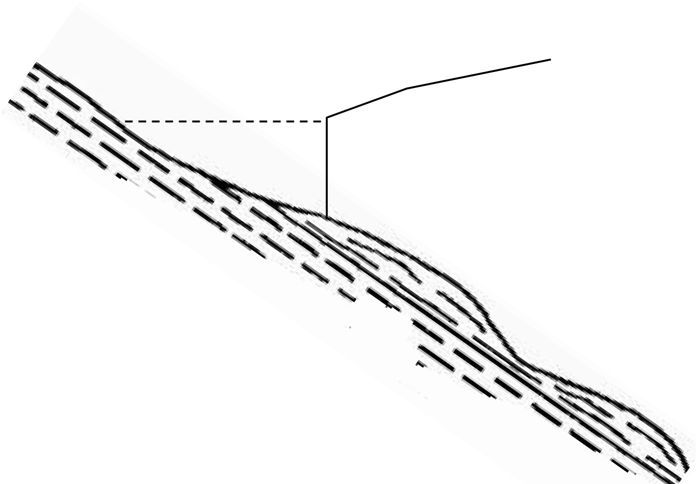
$$\overline{u}_k^{n+1} n_k^x + \overline{v}_k^{n+1} n_k^y + w_{i,k}^{n+1} n_k^z = \frac{\partial b}{\partial t}, \quad (k = k_b)$$

Surface b.c. not enforced (closure error)

Inundation algorithms

- Algorithm 1 (*inunfl*=0; default)
  - Update wet and dry elements, sides, and nodes at the end of each time step based on the newly computed elevations. Newly wetted vel., S,T etc calculated using average
- Algorithm 2 (*inunfl*=1; accurate inundation for wetting and drying with sufficient grid resolution)
  - Wet → add elements & extrapolate velocity
  - Dry → remove elements
  - Iterate until the new interface is found at step n+1
  - Extrapolate elevation at final interface (smoothing effects)





Extrapolation of surface

Balzano's case A

Radiation stress (Longuet-Higgins)

$$\frac{Du}{Dt} = \cdots - \frac{1}{\rho H}\frac{\partial H S_{xx}}{\partial x} - \frac{1}{\rho H}\frac{\partial H S_{xy}}{\partial y}$$

$$\frac{Dv}{Dt} = \cdots - \frac{1}{\rho H}\frac{\partial H S_{yy}}{\partial y} - \frac{1}{\rho H}\frac{\partial H S_{xy}}{\partial x}$$

S_{xx} etc are functions of wave energy, which is integrated over all freq. and direction

Radiation stress: vortex formalism

From Bennis and Ardhuin (2011)

$$\frac{\partial \hat{u}}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\partial \hat{u}}{\partial y} + \hat{w}\frac{\partial \hat{u}}{\partial z} - f\hat{v} + \frac{1}{\rho}\frac{\partial p^H}{\partial x} = \underbrace{\left[f + \left(\frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y}\right)\right]V_s - W_s\frac{\partial \hat{u}}{\partial z} - \frac{\partial J}{\partial x}}_{A1} + \underbrace{\widehat{F}_{m,x} + \widehat{F}_{d,x} + \widehat{F}_{b,x}}_{A2}, \quad (11)$$

and

$$\frac{\partial \hat{v}}{\partial t} + \hat{u}\frac{\partial \hat{v}}{\partial x} + \hat{v}\frac{\partial \hat{v}}{\partial y} + \hat{w}\frac{\partial \hat{v}}{\partial z} + f\hat{u} + \frac{1}{\rho}\frac{\partial p^H}{\partial y} = \underbrace{-\left[f + \left(\frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y}\right)\right]U_s - W_s\frac{\partial \hat{v}}{\partial z} - \frac{\partial J}{\partial y}}_{B1} + \underbrace{\widehat{F}_{m,y} + \widehat{F}_{d,y} + \widehat{F}_{b,y}}_{B2}, \quad (12)$$

$(\widehat{F}_{b,x}, \widehat{F}_{b,y})$ Sink due to wave bottom frition

(U_s, V_s, W_s) Stokes drift velocity

$(\widehat{F}_{d,x}, \widehat{F}_{d,y})$ Wave breaking/turbulence

J Wave–induced mean pressure

Also changes in the velocity land b.c. and in tracer transport

WBBL (Grant-Madsen)

Wave-current friction factor

Ratio of shear stresses

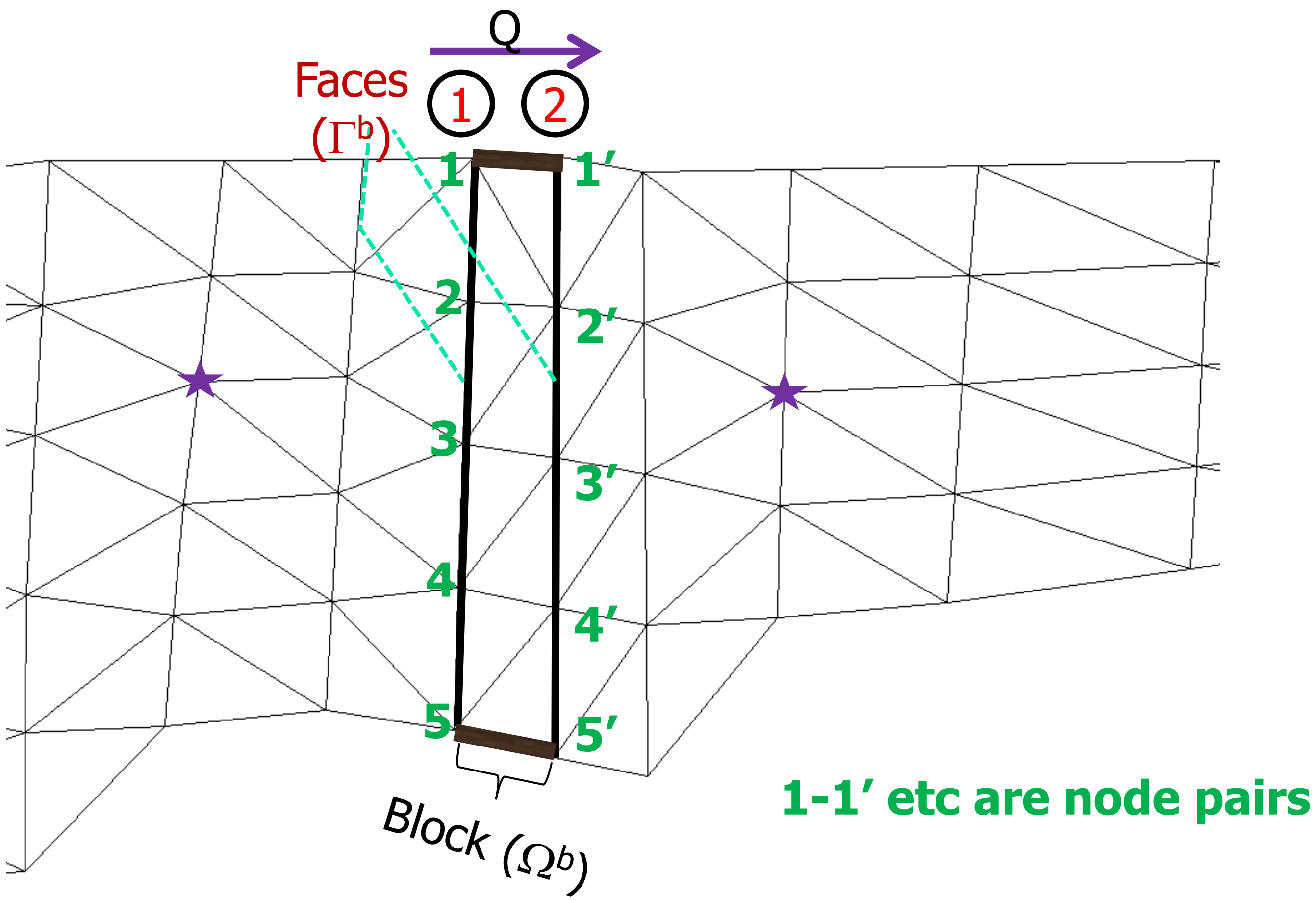
$$\begin{cases} \mu = \frac{\tau_b}{\tau_w} \\ c_\mu = (1 + 2\mu|\cos\varphi_{wc}| + \mu^2)^{1/2} \\ f_w = c_\mu exp[5.61(\frac{c_\mu U_w}{k_N \omega})^{-0.109} - 7.3] \\ \tau_w = 0.5 f_w U_w^2 \end{cases}$$

φ_{wc} angle between bottom current and dominant wave direction

τ_b: current induced bottom stress
U_w: orbital vel. amplitude
ω: representative angular freq.

The nonlinear eq. system is solved with a simple iterative scheme starting from μ=0 (pure wave), c_μ=1. Strong convergence is observed for practical applications

Hydraulics



The 2 'upstream' and 'downstream' reference nodes on each side of the structure are used to calculate the thru flow Q

Hydraulics

The differential-integral eq. for continuity eq. is modified as

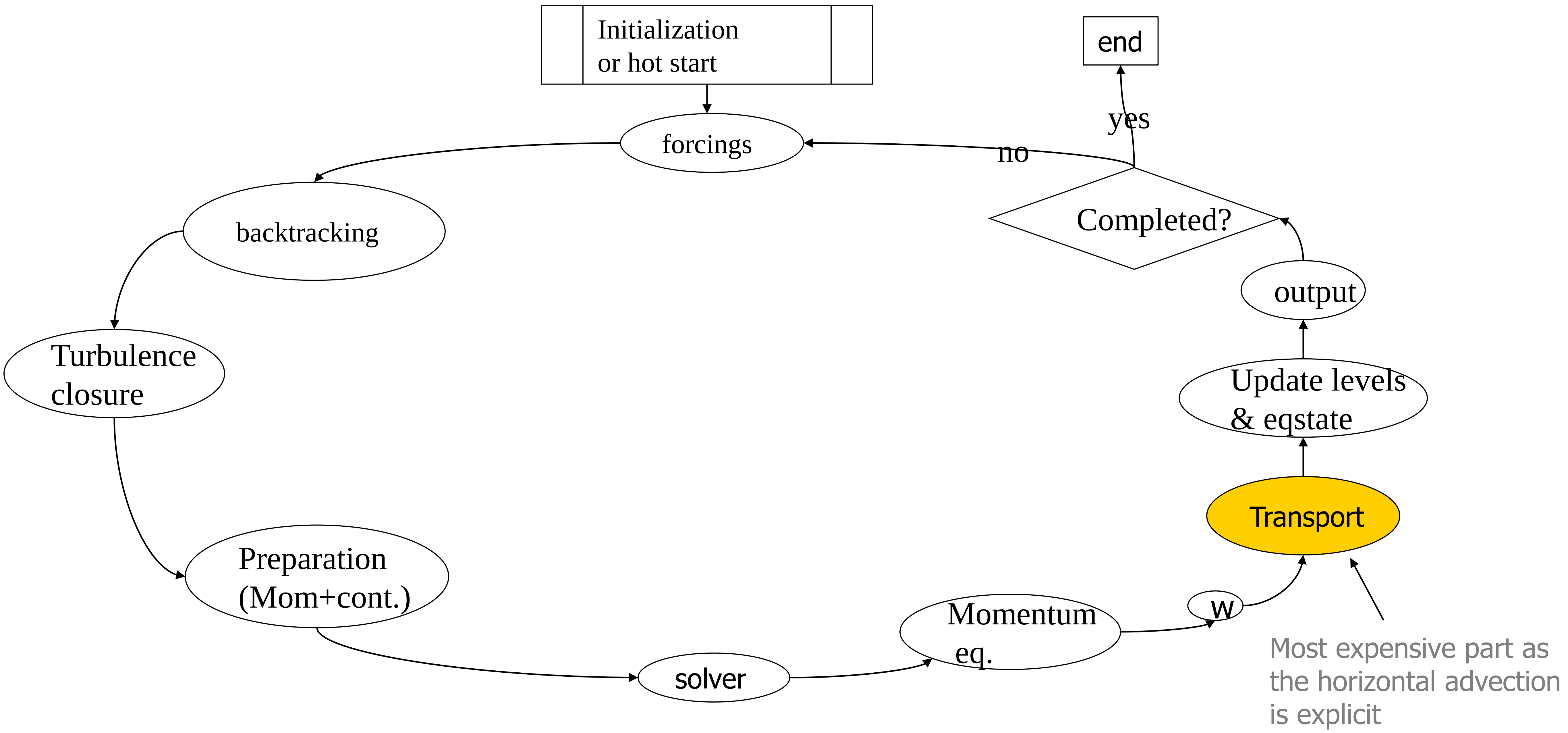
$$\int_{\Omega-\Omega^b}[\ldots]d\Omega = \int_{\Omega-\Omega^b}[\ldots]d\Omega - \theta\Delta t\int_{\Gamma\cup\Gamma^b}\phi_i\widehat{U}^{n+1}d\Gamma$$

$$-(1-\theta)\Delta t\int_{\Gamma\cup\Gamma^b}\phi_i U_n^n d\Gamma - \theta\Delta t\int_{\bar{\Gamma}_v}\phi_i U_n^{n+1}d\bar{\Gamma}_v$$

And for the momentum eq.

- Add Γ^b as a horizontal boundary in backtracking
- At Γ^b and also for all internal sides in Ω^b, the side vel. is imposed from Q – like an open bnd (add to I_3 and I_5)

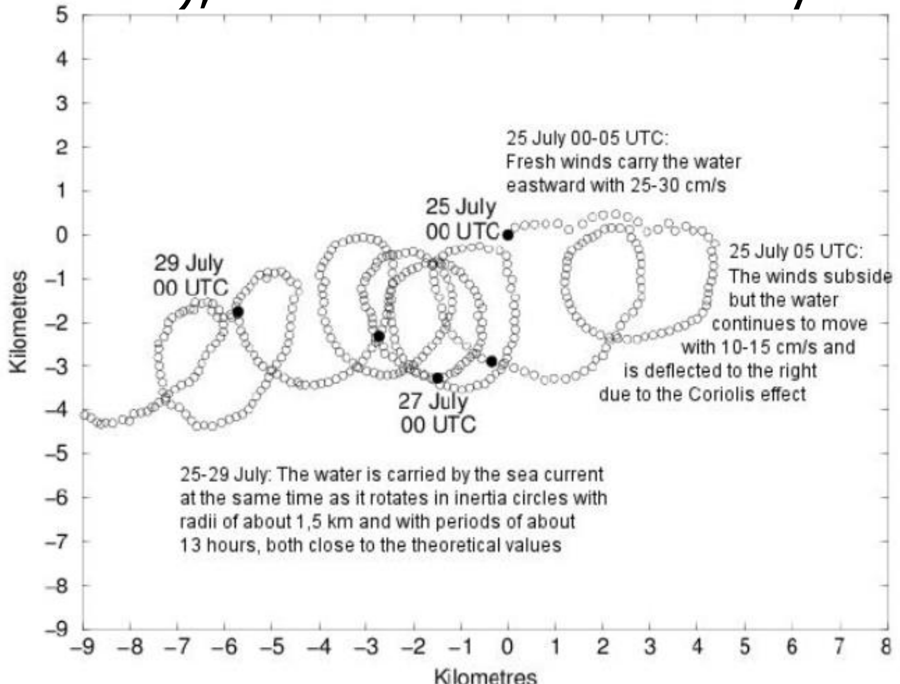
SCHISM flow chart



Numerical stability

- Semi-implicitness circumvents CFL (most severe) (Casulli and Cattani 1994)
- ELM bypasses Courant number condition for advection (actually CFL>0.4)
- Implicit TVD[2] transport scheme along the vertical bypasses Courant number condition in the vertical
- Explicit terms
  - Baroclinicity → internal Courant number restriction (usually not a problem) $\frac{\sqrt{g'h}\Delta t}{\Delta x} < 1$
  - Horizontal viscosity/diffusivity → diffusion number condition (mild) $\frac{\mu \Delta t}{\Delta x^2} \leq 0.5$
  - Upwind, TVD[2] transport or WEON[3]: horizontal Courant number condition → **sub-cycling**
  - Coriolis – "inertial modes" in eddying regime (Le Roux 2009; Danilov 2013); needs to be stabilized by viscosity/filter

$$2u_1^{n+1} - u_2^{n+1} - u_3^{n+1} = 0$$



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Lateral b.c. and nudging

bctides.in (Types 1–5, -1, -4/-5) | param.in (Nudging/sponge layer)

Variable	Type 1 (*.th)	Type 2:	Type 3	Type 4 (*[23]D.th)	Type 5	Type -1	Type -4, -5 (uv3D.th); nudging	Nudging/sponge layer near boundary
η	**elev.th**: Time history; uniform along bnd	constant	Tidal amp/phases	**elev2D.th.nc**: time- and space-varying along bnd	**elev2D.th.nc:** combination of 3&4	Must =0	N/A	inu_elev
S&T, tracers	**[MOD]_?.th** : relax to time history (uniform along bnd) for inflow	Relax to constant for inflow	Relax to i.c. for inflow	**[MOD]_3D.th.nc**: relax to time- and space-varying values along bnd during inflow	N/A	N/A	N/A	inu_[MOD]
u,v	**flux.th**: via discharge **(<0 for inflow!)**	Via discharge **(<0 for inflow!)**	Tidal amp/phases	**uv3D.th.nc**: time- and space-varying along bnd (in lon/lat for ics=2)	**uv3D.th.nc:** combination of 3&4	Flather ('0' for η)	Relax to **uv3D.th.nc** (2 separate relaxations for in & outflow)	inu_uv

4H's of SCHISM

- Hybrid FE-FV formulation
- Hybrid horizontal grid
- Hybrid vertical grid
- Hybrid code (openMP-MPI)